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ON THE ESTIMATION IN A CLASS OF DIFFUSION-TYPE PROCESSES. APPLICATION FOR DIFFUSION BRANCHING PROCESSES

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Introduction

In this work a stochastic differential equations family whose solutions are multidimensional diffusion-type (non necessarily markovians) processes is considered, and the estimation of a parametric vector θ which relations the coefficients is studied. The conditions for the existence of the likelihood function are proved and observing continuously the process, the estimator is obtained. An application for Diffusion Branching Processes is given. This problem has been studied in some special cases by Brown and Hewitt (1975), Liptser and Shiryayev (1978) and Sorensen (1983).

The basic model

Let (Ω,F,P) be our basic complete probability space, and let a sequence of σ-algebras $(F_t,\ 0\le t\le T<\infty)$ such that $s\le t \Rightarrow F_s \subseteq F_t \subseteq F$. We shall consider the stochastic differential equations family :

$$dX(t) = A(t,X)\theta dt + A^{1/2}(t,X)dW(t)$$
$$X(0) = Y \qquad 0\le t\le T \qquad (1)$$

where $(W(t),\ 0\le t\le T)$ is a n-dimensional Wiener process with independent components, $A(t,X)$ is a nonanticipating functional, Y is a random vector F_0-measurable such that $P(\sum|Y_i|<\infty)=1$, and θ is a parameter with values in an open set $\Theta\subseteq R^n$. Let us consider the case when the matrix $A(t,X)$ is know and non degenerate. Let C_T be the set of continuous functions f: $[0,T]\longrightarrow R^n$, and let $\beta_T = \sigma(x\colon x(s),\ 0\le s\le T)$. We shall denote by μ_θ the measure induced in (C_T,β_T) by the solution of (1) when θ is the true parameter.

Theorem

If for each $t\in[0,T]$ and $x,y\in C_T$ the components $A_{ij}(t,X)$, for $i,j=1,\dots,n$, of $A(t,X)$ satisfy the conditions :

i). $\quad A_{ij}^2(t,x) \le k_1\int_0^t(1+\sum x_i^2(s))dK(s) + k_2(1+\sum x_i^2(t))$

ii).

$$|A_{ij}(t,x)-A_{ij}(t,y)|^2 \le k_1\sum\int_0^t (x_i(s)-y_i(s))^2 dK(s) + k_2\sum(x_i(t)-y_i(t))^2$$

where k_1 and k_2 are constants, and K(.) is a nondecreasing right-continuous function, $0\le K(s)\le 1$. Then for all $\theta\epsilon\Theta$, $\mu_\theta\sim\mu_{\theta_0}$, where θ_0 denote a fixed value of the parameter, and the corresponding Radon-Nikodym derivative is :

$$\frac{d\mu_\theta}{d\mu_{\theta_0}}(X) = \exp\{(\theta-\theta_0)'(X(T)-Y) - (1/2)(\theta-\theta_0)'(\int_0^T A(t,X)dt)(\theta+\theta_0)\} \qquad (2)$$

proof

It is enough prove (Liptser and Shiryayev (1977) :

I). For i=1,...,n , the equation $A^{1/2}(t,X)B_i(t,X) = A_i(t,X)$ has solution with respect to $B_i(t,X)$ $[\mu_{\theta_0}]$, where $A_i(t,X)=(A_{1i}(t,X),\ldots,A_{ni}(t,X))'$

II). For i=1,...,n and $\theta\epsilon\Theta$ $\mu_\theta(\int_0^T B_i'(t,X)B_i(t,X)dt<\infty)=1$

But since $A^{1/2}(t,X)$ is a non-singular matrix and $B_i'(t,X)B_i(t,X)= A_i'(t,X)A^{-1}(t,X)A_i(t,X) = A_{ii}(t,X)$, I) and II) are hold.

Maximum Likelihood Estimation.

Suppose we observe a process X , which we know solves one of the equations (1) continuously in the time interval [0,T] and that we want to infer which one it is. For this purpose let us consider the likelihood function $L_T(\theta) = (d\mu_\theta/d\mu_{\theta_0})(X)$. From (2) we find that :

$$\dot{l}_T(\theta) = (X(T)-Y) - (\int_0^T A(t,X)dt)\theta \quad \text{and} \quad \ddot{l}_T(\theta) = -\int_0^T A(t,X)dt \qquad (3)$$

where $l_T(\theta) = \ln(L_T(\theta))$ and "." denote derivative with respect to θ.

The solution of the likelihood equation $\dot{l}_T(\theta) = 0$ is :

$$\hat{\theta}_T = (X(T)-Y)(\int_0^T A(t,X)dt)^{-1}$$

If $\hat{\theta}_T\in\Theta$, from (3) it is the unique maximum likelihood estimate.

The observed Fisher information is $\int_0^T A(t,X)dt$, and the statistic $(X(T)-Y , \int_0^T A(t,X)dt)$ is sufficient for θ.

Application for Diffusion Branching Processes.

A Diffusion Branching Process (DBP), is a non negative diffusion process with $X(0) = a>0$, drift coefficient θx, and diffusion coefficient αx, $(\alpha > 0)$, where a, θ, and α are constants. The DBP serve in their own right as models for various physical and biological phenomena, but are probably viewed most often as approximations to Galton Watson processes. Inference on a DBP will be inference on the two parameters θ and α. However, provided we observed X in continuous time, we can use a version of the Lévy result for the quadratic variation of Brownian Motion to give us α, (Basawa and Prakasa-Rao (1980), pp:212). Therefore without loss of generality, take $\alpha=1$. The estimation of θ, has been considered using a sequential procedure by Brown and Hewitt (1975). But the DBP with drift coefficient θx, $(\theta \in R)$, diffusion coefficient x, and initial value $X(0)=a$, are solutions of the stochastic differential equations family :

$$dX(t) = \theta X(t)dt + \sqrt{X(t)}dW(t)$$
$$X(0) = a \qquad 0 \le t \le T \qquad (4)$$

But (4) is the particular case of (1) when $n=1$, $A(t,X)=X(t)$, $Y=a$. Therefore from (2) the Radon-Nikodym derivative is :

$$\frac{d\mu_\theta}{d\mu_{\theta_0}}(X) = \exp\{(\theta - \theta_0)(X(T)-a) - (1/2)(\theta^2 - \theta_0^2)\int_0^T X(t)dt\}$$

and the estimator of θ is $\hat{\theta}_T = (X(T)-a)/(\int_0^T X(t)dt)$. A sufficient statistic for θ is $(X(T), \int_0^T X(t)dt)$, and $\int_0^T X(t)dt$ is the observed Fisher information.

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