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Properties and Applications of Tauberian Operators

Manuel González

Departamento de Matemáticas, Facultad de Ciencias, Universidad de Cantabria, 39071 Santander, Spain.

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Dedicated to the memory of
José María García Lafuente.

ABSTRACT

Tauberian operators, which appeared in reponse to a problem in summability [GaW, KW], have found application in several situations: factorization of operators [DFJP], preservation of isomorphic properties of Banach spaces [N, NR], equivalence between the Radon-Nikodym property and the Krein-Milman property [Sch], and generalized Fredholm operators [Ta, Y].

This paper is a survey of the main properties and applications of tauberian operators.

0. INTRODUCTION

Tauberian operators were introduced by Kalton and Wilansky as (continuous linear) operators on Banach spaces $T : X \longrightarrow Y$ such that $T^{**-1}Y = X$, where T^{**} is the second conjugate of T. This property is an abstract version of a property of certain conservative matrices, considered by Garling and Wilansky [GaW], and other authors. Moreover, the inclusion map in the factorization theorem of [DFJP] was constructed verifying it. In fact, it has been shown [N] that most of the results about the intermediate space of the factorization can be derived from the fact that this inclusion map is tauberian.

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Moreover, tauberian operators can be seen as generalized upper semi-Fredholm operators associated with the class of reflexive spaces [Y].

In this paper we are going to describe the main results and applications of tauberian operators that can be found in the literature.

The paper is organized as follows:

Section 1 contains the definition and basic properties of tauberian operators, including the characterizations by the action over relatively weakly non-compact, bounded subsets (or sequences). We also present some results relating tauberian operators and upper semi-Fredholm operators.

In section 2 we give several examples of tauberian operators, showing the situations in which these operators appear. They also will be used as counterexamples in other sections of the paper.

In section 3 we consider the main reason for the interest of studying tauberian operators, which is the preservation of isomorphic properties of (bounded subsets of) Banach spaces.

Section 4 includes two interesting characterizations of tauberian operators. One of them in terms of perturbations by compact operators, and the other by the action over closed convex bounded subsets. We give also a construction of an injective tauberian operator $j : \ell_2(X) \longrightarrow X$ beginning from an injective tauberian operator $i: X\times X \longrightarrow X$, and we show some applications of the characterizations.

Finally, in section 5, we consider briefly a dual class, the cotauberian operators, and other classes with analogous properties to that of tauberian and cotauberian operators. Also we show other situations in which definitions of tauberian operators have been considered.

<u>Notations</u>:

X, Y, Z will denote Banach spaces, $B_X := \{x\in X : \|x\|\leq 1\}$ the unit ball of X, X^* the dual space of X, and L(X,Y) the class of all (continuous linear) operators acting from X into Y. Given $T \in L(X,Y)$, we shall denote by R(T) and N(T) the range and the kernel of T, $T^* \in L(Y^*,X^*)$ will be the conjugate operator of T. Finally, we shall say that T is *upper semi-Fredholm* if R(T) is closed and N(T) is finite dimensional.

1. DEFINITION AND FIRST PROPERTIES.

In this section we include the basic properties of tauberian operators, as can be found in [KW], and other results obtained in [GO1]. Also we give the characterization of these operators by its action over relatively weakly non-compact subsets, and show the relation between tauberian operators and upper semi-Fredholm operators.

1.1 Definition An operator $T : X \longrightarrow Y$ is *tauberian* if $T^{**^{-1}}Y = X$.

Clearly $T : X \longrightarrow Y$ is tauberian if and only if the associated operator $T_q : X^{**}/X \longrightarrow Y^{**}/Y$ given by

$$T_q(F+X) := T^{**}F+Y$$

is injective [Y].

Note also that the *weakly compact* operators are defined by an exactly opposite property: $T^{**^{-1}}Y = X^{**}$.

1.2 Proposition [KW] Given an operator $T : X \longrightarrow Y$ we have

(a) T tauberian $\Longrightarrow$ $N(T^{**}) \subset X$ $\Longrightarrow$ $N(T)$ reflexive.

For range closed operators the converse implications are true.

(b) T tauberian and $K : X \longrightarrow Y$ weakly compact $\Longrightarrow$ $T+K$ tauberian.

The following result is an immediate consequence of the definition.

1.3 Proposition [KW] For an operator $T : X \longrightarrow Y$ the following assertions are equivalent:

(a) T is tauberian.

(b) TB_X is closed and $N(T^{**}) \subset X$.

(c) $\overline{T(B_X)} \subset R(T)$ and $N(T^{**}) \subset X$.

Using the Eberlein-Smulian theorem we can prove the following theorem.

1.4 Theorem [KW] An operator $T : X \longrightarrow Y$ is tauberian if and only if for every bounded subset $A \subset X$ such that TA is relatively (weakly) compact we have that A is relatively weakly compact.

Observation The above result shows that tauberian operators preserve the relative weak non-compactness of bounded sets. These operators also preserve many other isomorphic properties of subsets or spaces. This topic, which has been systematically studied in [N], will be treated in section 3.

It follows also from the Eberlein-Smulian theorem that we can write the above characterization in terms of sequences.

1.5 Theorem An operator $T : X \longrightarrow Y$ is tauberian if and only if for every bounded sequence $(x_n) \subset X$ such that (Tx_n) is (weakly) convergent we have that (x_n) has a weakly convergent subsequence.

From this theorem we derive the following algebraic characterization.

1.6 Corollary [GO3] An operator $T : X \longrightarrow Y$ is tauberian if and only if given Z and $A : Z \longrightarrow X$, TA weakly compact implies A weakly compact.

Proof. We will give an easier proof than that in [GO3].

Suppose T is tauberian. If TA is weakly compact, then we have $(TA)^{**^{-1}}Y = A^{**^{-1}}T^{**^{-1}}Y = A^{**^{-1}}X = Z^{**}$; hence A is weakly compact.

Conversely, if T is not tauberian, then there exists a bounded sequence (x_n) in X having no weakly convergent subsequences such that (Tx_n) is weakly convergent.

Now, if we consider the operator $A : \ell_1 \longrightarrow X$ defined by $Ae_n := x_n$, where e_n denotes the unit vector basis in ℓ_1, we have that TA is weakly compact, but A is not weakly compact, since $\overline{AB_{\ell_1}}$ coincides with the absolutely convex, closed hull of $\{Ae_n\}$.

Next, we present a characterization of upper semi-Fredholm operators and very irreflexive Banach spaces in terms of tauberian operators.

1.7 Theorem [GO1] (a) $T : X \longrightarrow Y$ is upper semi-Fredholm if and only if it is tauberian and the restrictions of T to reflexive subspaces of X are upper semi-Fredholm.

(b) A Banach space X contains no reflexive, infinite dimensional subspaces if and only if every tauberian operator with domain in X is upper semi-Fredholm.

Finally, we show that tauberian operators can be seen as generalized upper semi-Fredholm operators associated with the class of reflexive spaces.

1.8 Proposition [AG, GO1, Y] Let $T, S : X \longrightarrow Y$ and $S : Y \longrightarrow Z$ be operators. Then we have

(a) S, T tauberian $\Longrightarrow$ ST tauberian $\Longrightarrow$ T tauberian.

(b) T tauberian, K weakly compact $\Longrightarrow$ $T+K$ tauberian.

(c) T tauberian $\Longrightarrow$ $N(T)$ reflexive.

(d) Suppose $R(T)$ closed. Then T tauberian $\Longleftrightarrow$ $N(T)$ reflexive.

(e) Tauberian operators with closed range belong to the (topological) interior of the class of all tauberian operators.

1.9 Observation [AG] In the above result we showed some properties which are shared by tauberian and upper semi-Fredholm operators. However, there are also properties that distinguish these two classes:

(a) T^{**} tauberian $\Longrightarrow$ T tauberian, but the converse is not true. (See example 2.C).

(b) $\{ T \in L(X,Y) : T \text{ tauberian} \}$ is not always open in $L(X,Y)$. (See example 2.F).

2. EXAMPLES

This section includes several examples that give an idea of the situations in which tauberian operators appear. Also they will be used as counterexamples in other sections.

(A) Isomorphisms (more generally, upper semi-Fredholm operators) are trivial examples of tauberian operators.

(B) The main source of non-trivial examples of tauberian operators is the well-known factorization of operators introduced in [DFJP].

Given an operator $T : X \longrightarrow Y$, a canonical construction provides us with a Banach space Z and operators $A : X \longrightarrow Z$ and $j : Z \longrightarrow Y$ in such a form that $T = jA$.

j is a tauberian operator [DFJP].

Moreover, all the conjugates of even order $j^{(2n)}$ of j are tauberian operators [AG]. Next example shows that this is not true for arbitrary tauberian operators.

(C) There exists a Banach space Z and a tauberian operator $T : Z \longrightarrow Z$ such that T^{**} is not tauberian [AG].

The idea of the construction of this example is as follows:

Recall that $T : X \longrightarrow Y$ is tauberian if and only if the associated operator $T_q : X^{**}/X \longrightarrow Y^{**}/Y$ is injective [Y].

Using a construction due to Bellenot [Bel] we construct an space Z such that $Z^{**}/Z \simeq \ell_1$. Next we construct an operator $T : Z \longrightarrow Z$ such that T_q can be identified with $A : \ell_1 \longrightarrow \ell_1$ given by $A(x_n) := (x_n/n)$, and $(T^{**})_q$ can be identified with A^{**}.

Clearly T is tauberian, since A is injective, and T^{**} is not tauberian, since A^{**} is not injective.

(D) Let E, F be an interpolation pair of Banach spaces, $1<p<\infty$, and $0<\theta<1$. Then the injection of the real interpolation space $(E,F)_{\theta,p}$ into $E+F$ is a tauberian operator [Bea, II.2.Prop. 1].

(E) Let X be a Banach space, and let ϕ be a Young function such that ϕ and its conjugate verify the Δ_2-condition. Then the natural inclusion of the Orlicz space of vector valued functions $L_\phi(X)$ into $L_1(X)$ is tauberian [BoFi].

This result was applied in [Bo] to study the (V*) sets in $L_\phi(X)$ by means of the properties of the (V*) sets in $L_1(X)$.

(F) Let X be a non-reflexive Banach space. The operator

$$T : \ell_2(X) \longrightarrow \ell_2(X)$$

defined by $T(x_n) := (x_n/n)$ is a tauberian operator.

However, the operators $T_k : \ell_2(X) \longrightarrow \ell_2(X)$ defined by

$$T_k(x_n) := (x_1, x_2/2, \ldots, x_k/k, 0, 0, \ldots)$$

are not tauberian, since its kernel is not reflexive, and $\|T-T_k\| = 1/k$.

In general, given a bounded sequence $S_n : X_n \longrightarrow Y_n$ of tauberian operators, the operator

$$T : (x_n) \in \ell(X_n) \longrightarrow (S_n x_n) \in \ell(Y_n)$$

is tauberian.

(G) Let J be the James quasireflexive space, as defined for example in [LT; 1.d.2]. We have

$$J := \{ (a_n) \in c_o \ / \ \|(a_n)\|_J < \infty \}$$

where

$$\|(a_n)\|_J := \sup \left\{ \left(\sum_{l=1}^{m-1} (a_{p_l} - a_{p_{l+1}})^2 \right)^{1/2} : m \in \mathbb{N},\ p_1 < p_2 < \dots < p_m \right\}.$$

The natural inclusion of J into c_o is a tauberian operator.

This is a direct consequence of the fact that J^{**} can be expressed as the direct sum of J and the subspace generated by the sequence $(1,1,1,\dots)$, which does not belong to c_o.

3. PRESERVATION OF ISOMORPHIC PROPERTIES

A useful tool in the isomorphic theory of Banach spaces is the study of operators which are less restrictive than isomorphisms but yet preserve some of the space (or subset) isomorphic properties. This tool has been developed quite recently.

In 1974 Davis et al. [DFJP] developed a construction technique in which, for a bounded subset W of a Banach space X, a new Banach space Y and an operator $J : Y \longrightarrow X$ are constructed. This was used to show that every weakly compact operator factors through a reflexive space, and has also been used to construct many counterexamples.

One reason this construction works is that the operator J is not an isomorphism, but preserves many properties including the relative weak compactness. In [KW], Kalton and Wilansky isolated this type of operator and defined it to be tauberian.

In this section we will review the properties preserved by tauberian operators. For a systematic study of this topic we refer to [N], where other classes, like semiembeddings, G_δ-embeddings, ... are considered.

First we recall some definitions.

3.1 Definition Let X be a Banach space.

X is *quasi-reflexive* if $\dim X^{**}/X < \infty$.

X is *somewhat reflexive* if every infinite dimensional subspace of X contains an infinite dimensional reflexive subspace.

X has *property (u)* if for every weakly Cauchy sequence $(x_n) \subset X$, there exists a weakly unconditionally Cauchy series $\sum y_k$ in X such that the sequence $\left(x_n - \sum_{k=1}^{n} y_k\right)$ is weakly null.

3.2 Theorem [N] Let X, Y be Banach spaces, and suppose that there exists a tauberian operator $T : X \longrightarrow Y$.

If Y has one of the following properties, then X has the same property:

(a) Reflexivity, quasi-reflexivity, somewhat reflexivity.

(b) Weak sequential completeness.

(c) ℓ_1 does not embed in Y.

(d) c_o does not embed in Y.

(e) Radon-Nikodym property.

3.3 Observation Example (G) is section 2 shows that property (u) is not preserved by tauberian operators, since c_o has property (u), but J has not this property.

Part of the results in the above theorem can be "localized", as we show in the following theorem.

3.4 Theorem [N] Let $T : X \longrightarrow Y$ be a tauberian operator, and let A be a bounded subset of X.

If TA has one of the following properties, then A has the same property:

(a') relative weak compactness: each sequence in TA has a weakly convergent subsequence.

(b') weak completeness: each weak Cauchy sequence in TA is weakly convergent.

(c') weak precompactness: each sequence in TA has a weakly Cauchy subsequence.

(d') weak unconditional completeness: each weakly unconditionally Cauchy series in TA is unconditionally convergent.

The above results 3.2 and 3.4 can be applied to study the intermediate space of the factorization of operators given in [DFJP] and the real interpolation spaces [Bea].

4. CHARACTERIZATIONS AND APPLICATIONS

In this section we present two important characterizations of tauberian operators and some applications.

First we give a perturbative characterization which is analogous to another one for upper semi-Fredholm operators

4.1 Theorem [GO3] An operator $T : X \longrightarrow Y$ is tauberian if and only if $N(T+K)$ is reflexive for every compact operator $K : X \longrightarrow Y$.

In fact, if T is not tauberian, then we can find a non-reflexive subspace M of X such that the restriction of T to M is nuclear with arbitrarily small norm.

In [Hm], Herman introduced the class of *almost weakly compact* (a.w.c.) operators as those operators which have no bounded inverses in non-reflexive subspaces. This definition generalizes the stricly singular operators, allowing reflexive subspaces instead of finite dimensional subspaces. However, Herman was not able to decide if the sum of two a.w.c. operators is also an a.w.c. operator.

The above perturbative characterization allowed to show [G1] that the ℝ-strictly singular operators, defined below, are the right extension of the strictly singular operators to the reflexive setting.

4.2 Definition [G1] An operator $T : X \longrightarrow Y$ is *ℝ-strictly singular* if given $A : Z \longrightarrow X$ we have that TA tauberian $\Longrightarrow$ A weakly compact.

4.3 Proposition [G1] The class of ℝ-strictly singular operators is an operator ideal (in the sense of [Pi]) which contains properly the strictly singular and the weakly compact operators.

Next we give some characterizations of tauberian operators, due to Neidinger and Rosenthal, in terms of the action over closed convex bounded subsets. We observe that an operator is upper semi-Fredholm if and only if it takes closed bounded sets into closed sets.

4.4 Theorem [NR] For a non-zero operator $T : X \longrightarrow Y$ the following properties are equivalent:

(a) T is tauberian.

(b) For all closed convex bounded subsets K of X, TK is closed.

(c) For every closed subspace M of X, TB_M is closed.

(d) For every closed subspace M of X, $\overline{TB_M} \subset R(T)$.

4.5 Observation A linear functional on a non-reflexive space X is not tauberian. Thus (b) in the above theorem tells us that for every $f \in X^*$ we can construct a closed subspace M of X such that $f(B_M)$ is not closed; i.e., the restriction $f|_M$ does not attain its norm.

This result complements James' characterization: If (and only if) the space X is non-reflexive, then there exists an $f \in X^*$ which does not attain its norm on B_X. In fact the proof in [NR] uses the full generality of that result.

We saw in section 3 (example F) that the class of tauberian operators is not open. This is a shortcoming from the point of view of perturbation theory, but it allows the following result, which is very interesting from the point of view of the preservation of isomorphic properties.

This result was proved in [Sch] for injective operators taking bounded, closed convex subsets into closed sets (cfr. (b) in the above theorem). We give here a more elementary proof.

4.6 Proposition [Sch] Given an injective tauberian operator $i : X \times X \longrightarrow X$ we can construct an injective tauberian operator $j : \ell_2(X) \longrightarrow X$.

Proof. We may assume $r := \|i\| < 1$, and $i(x,y) := u(x) + v(y)$, with $u, v : X \longrightarrow X$, $\|u\|, \|v\| \le r$.

Then, for a finitely supported sequence $(x_1, x_2, \ldots, x_n, 0, 0, \ldots) \in \ell_2(X)$ we define $j(x_k)$ recursively by

$$j(0,0,0,\ldots) := 0; \qquad j(x_k) := i(x_1, j(x_{k+1})).$$

We have

$$j(x_1, x_2, \ldots, x_n, 0, 0, \ldots) = \sum_{i=1}^{n} v^{i-1} u x_i .$$

Clearly j is linear. Also we have

$$\| j(x_k) \| \le \sum_{i=1}^{n} r^i \|x_i\| \le \left(\sum_{i=1}^{n} r^{2i} \right)^{1/2} \left(\sum_{i=1}^{n} \|x_i\|^2 \right)^{1/2} .$$

Hence j is continuous and can be extended to all the space $\ell_2(X)$.

Since i is injective, and

$$j(x_k) = i(x_1, j(x_{k+1})) \quad \text{for every} \quad (x_k) \in \ell_2(X)$$

it follows that j is injective:

$$j(x_k) = 0 \Longrightarrow x_1 = 0,\ j(x_{k+1}) = 0 \Longrightarrow x_2 = 0,\ j(x_{k+2}) = 0 \Longrightarrow \dots$$

Moreover, since we have the same relation for the second conjugates

$$j^{**}(z_k) = i^{**}(z_1, j^{**}(z_{k+1})) \quad \text{for every} \quad (z_k) \in \ell_2(X^{**}),$$

and i is tauberian, we have that j is tauberian:

$$j^{**}(z_k) \in X \Longrightarrow z_1 \in X^{**},\ j^{**}(z_{k+1}) \in X \Longrightarrow z_2 \in X,\ j^{**}(z_{k+2}) \in X \Longrightarrow \dots$$

Using the construction in the above proposition, Schachermayer proved the equivalence between the Radon-Nikodym property (RNP) and the Krein-Milman property (KMP) for a certain class of Banach spaces.

4.7 Theorem [Sch] Let X be a Banach space, and suppose that there exists an injective tauberian operator $i : X \times X \longrightarrow X$.

X has the RNP if and only if it has the KMP.

5 EXTENSIONS AND GENERALIZATIONS

We describe briefly certain classes of operators related to tauberian operators which have been considered in the literature.

Cotauberian operators

An operator $T : X \longrightarrow Y$ is said to be *cotauberian* if its conjugate T^* is tauberian.

Considering the associated operator $T_q : X^{**}/X \longrightarrow Y^{**}/Y$, we have that T is cotauberian if and only if the range of T_q is dense.

The properties of cotauberian are dual than that of tauberian operators (see [G1, GO2, GO3, Ta, Y]). As an example we give the following result:

5.2 Theorem [GO3] For an operator $T : X \longrightarrow Y$ the following properties are equivalent:

(a) T is cotauberian.

(b) For every compact operator $K : X \longrightarrow Y$, $Y/\overline{R(T+K)}$ is reflexive.

(c) Given $B \in L(Y,Z)$, BT weakly compact $\Longrightarrow$ BT is weakly compact.

"Sequential" semigroups

The operator ideals of compact, weakly compact, Rosenthal, completely continuous and weakly completely continuous operators can be characterized by means of sequences [GO1] (see also [Pi]).

In [GO1, GO2, GO3] there were introduced and studied two semigroups asociated to each of the above mentioned operator ideals. For the weakly compact operators we have that the semigroups are the classes of tauberian and cotauberian operators, respectively.

All these semigroups have properties formally analogous to that of tauberian operators, and they verify the same relations of inclusion that the associated operator ideals.

One of the semigroups associated to the Rosenthal operators has been studied also in [MaS] (operators preserving mere weakly Cauchy sequences) and in [Hn] (semitauberian operators).

"Algebraic" semigroups

Given an operator ideal U, it is possible to introduce two semigroups SU_+ and SU_- in the following way [G]:

$$SU_+(X,Y) := \{ T\in L(X,Y) : A\in L(Z,X),\ TA\in U \Rightarrow A\in U \}$$

$$SU_-(X,Y) := \{ T\in L(X,Y) : B\in L(Y,Z),\ BT\in U \Rightarrow B\in U \}$$

For U the weakly compact operators, we obtain the tauberian and cotauberian operators, respectively.

"Metric" semigroups

Extending certain characterizations of upper and lower semi-Fredholm operators, there have been introduced in [GM, M] generalized semi-Fredholm operators associated to an space ideal by means of suitable operator quantities related with the norm.

The semigroups associated to the space ideal of reflexive spaces are contained in the classes of tauberian and cotauberian operators, respectively. In general, these semigroups are smaller than the "algebraic" ones.

"Ideal variation" semigroups

In [AsT] it has been introduced a semigroup of operators, included in the class of tauberian operators, by means of the weak measure of non-compactness. This procedure can be extended to obtain a semigroup associated to any operator ideal, using the ideal variation of [As].

Unbounded tauberian operators

In [Cr1, Cr2] the definition of tauberian operator is extended to the case of linear (not necessarily continuous) operators acting between normed spaces. Part of the results of [KW] are extended to this situation.

Non-archimedian tauberian operators

In [MmPe] the definition of tauberian operator is analyzed in the context of continuous operators in non-archimedian Banach spaces. It is shown (with some restrictions) that the class of tauberian operators coincides with the class of upper semi-Fredholm operators.

REFERENCES

[AG] Alvarez, T.; González, M. "Some examples of tauberian operators". Proc. Amer. Math. Soc. (to appear). EXTRACTA MATH. 4.2 (1989).

[As] Astala, K. "On measures of noncompactness and ideal variations in Banach spaces". Ann Acad. Sci. Fenn. Ser. A. I. Math. Diss 29 (1980), 1-42.

[AsT] Astala, K.; Tylli, H.-O. "Seminorms related to weak compactness and to tauberian operators". Math. Proc. Cambridge Phil. Soc. 107 (1990), 367-375.

[Bea] Beauzamy, B. Espaces d'interpolation réels: Topologie et géométrie". Springer Lecture Notes in Math. 666 (1978).

[Bel] Bellenot, S. "The J-sum of Banach spaces". J. Funct. Anal. 48 (1982), 95-106.

[Bo] Bombal, F. "On (V*) sets and Pelczynski's property (V*)". Glasgow Math. J. 32 (1990), 109-120.

[BoFi] Bombal, F.; Fierro, C. "Compacidad débil en espacios de Orlicz de funciones vectoriales". Rev. Real Acad. Ciencias Madrid 78 (1984), 157-163.

[Cr1] Cross, R.W. "Linear transformations of tauberian type in normed spaces". Volume dedicated to the memory of Prof. G. Köthe (to appear).

[Cr2] Cross, R.W. "F_+-operators are tauberian". Preprint Dept. Math. Univ. Cape Town 99/1990.

[DFJP] Davis, W.J.; Figiel, T.; Johnson, W.B. and Pelczynski, A. "Factoring weakly compact operators". J. Funct. Anal. 17 (1974), 311-327.

[GaW] Garling, D.J.H.; Wilansky, A. "On a summability theorem of Berg, Crawford and Whitley". Proc. Cambridge Phil. Soc. 71 (1972), 495-497.

[G1] González, M. "An operator ideal associated to tauberian operators". Preprint, 1990. EXTRACTA MATH. 4.2 (1989).

[G2] González, M. "Ideal semigroups associated to an operator ideal". Preprint Dep. Mat. Univ. Cantabria 2/1989. EXTRACTA MATH. 4.1 (1989).

[GM] González, M.; Martinón, A. "A generalization of semi-Fredholm operators". Preprint Dep. Mat. Univ. Cantabria 7/1990. EXTRACTA MATH. 5.2 (1990).

[GO1] González, M. and Onieva, V.M. "Semi-Fredholm operators and semigroups associated with some classical operator ideals". Proc. R. Ir. Acad. 88A (1988), 35-38. EXTRACTA MATH. 3.1 (1988).

[GO2] González, M. and Onieva, V.M. "Semi-Fredholm operators and semigroups associated with some classical operator ideals II". Proc. R. Ir. Acad. 88A (1988), 119-124. EXTRACTA MATH. 3.2 (1988).

[GO3] González, M., Onieva, V.M: "Characterizations of tauberian operators and other semigroups of operators". Proc. Amer. Math. Soc. 108 (1990), 399-405. EXTRACTA MATH. 3.3 (1988).

[Hm] Herman, R.H: "Generalizations of weakly compact operators" Trans. Amer. Math. Soc. 132 (1968), 377-386.

[Hn] Hernando, B. "Some new classes of operators between Banach spaces. Semitauberian operators". Preprint 1990.

[KW] Kalton, N.J. and Wilansky, A. "Tauberian operators in Banach spaces". Proc. Amer. Math. Soc. 57 (1976), 251-255.

[LT] Lindenstrauss, J.; Tzafriri, L. "Classical Banach spaces I. Sequence spaces". Springer 1977.

[MaS] Martin, D.H.; Swart, J. "A characterization of semi-Fredholm operators defined on almost reflexive Banach spaces". Proc. R. Ir. Acad. 86A (1986), 91-93.

[MmPe] Martínez Maurica, J.; Pellón, T. "Non-archimedian tauberian operators". Proc. Conf. on p-adic analysis, Hengelhoef 1986, 101-111.

[M] Martinón, A. "Cantidades operacionales en teoría de Fredholm". Ph. D. Thesis. University of La Laguna, 1989.

[N] Neidinger, R.D. "Properties of tauberian operators on Banach spaces". Ph. D. Thesis. University of Texas at Austin, 1984.

[NR] Neidinger, R.; Rosenthal, H.P. "Norm-attainment of linear functionals on subspaces and characterizations of tauberian operators". Pacific J. Math. 118 (1985), 39-339.

[Pi] Pietsch, A. "Operator ideals". North-Holland, Amsterdam, 1980.

[Sch] Schachermayer, W. "For a Banach space isomorphic to its square the Radon-Nikodym property and the Krein-Milman property are equivalent". Studia Math. 81 (1985), 329-339.

[Ta] Tacon, D.G. "Generalized semi-Fredholm transformations". J. Austral. Math. Soc. A34 (1983), 60-70.

[Y] Yang, K.W. "The generalized Fredholm operators". Trans. Amer. Math. Soc. 219 (1976), 313-326.