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WHITNEY'S EXTENSION THEOREM FOR NON-QUASI-ANALYTIC CLASSES OF ULTRADIFFERENTIABLE FUNCTIONS

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This note can be considered as a long summary of the invited lecture given by J. Bonet in the Second Functional Analysis Meeting held in Jarandilla de la Vega (Cáceres) in June 1990 and it is based on our joint article [2], which will appear in Studia Mathematica.

Various versions of Whitney's extension theorem and of E. Borel's theorem for different classes of ultradifferentiable functions have been given by many authors. We refer to our article [2] for precise references. In the present paper we use the classes $\mathcal{E}_{(\omega)}$ and $\mathcal{E}_{\{\omega\}}$ whose definition we recall immediately.

Modifying an idea of Beurling; Braun, Meise and Taylor [3] have shown that reasonable classes of non-quasianalytic functions can be introduced as follows. A continuous function $\omega: [0,\infty[\longrightarrow [0,\infty[$ is called a *weight function* if it satisfies

(α) $\omega(2t)=O(\omega(t))$ $\qquad$ (β) $\int_1^\infty \frac{\omega(t)}{t^2}\,dt < \infty$

(γ) $\log t = o(w(t)$ $\qquad$ (δ) $\varphi\colon t \longrightarrow \omega(e^t)$ is convex

Let $\varphi^*(y):= \sup\{xy-\varphi(x) : x\geq 0\}$, for $y>0$, be the Young conjugate of φ. For an open set $\Omega \neq \emptyset$ of $\mathbb{R}^N$ we define the spaces

$$\mathcal{E}_{\{\omega\}}(\Omega):= \{f\in C^\infty(\Omega): \forall K\subset\subset\Omega\ \exists m\in\mathbb{N} : \sup_{x\in K} \sup_{\alpha\in\mathbb{N}_o^N} |f^{(\alpha)}(x)|\ \exp(-\tfrac{1}{m}\varphi^*(m|\alpha|))<\infty\}$$

$$\mathcal{E}_{(\omega)}(\Omega):= \{f\in C^\infty(\Omega): \forall K\subset\subset\Omega\ \forall m\in\mathbb{N} : \sup_{x\in K} \sup_{\alpha\in\mathbb{N}_o^N} |f^{(\alpha)}(x)|\ \exp(-m\,\varphi^*(\tfrac{1}{m}|\alpha|))<\infty\}$$

These classes are non-quasianalytic for each weight function ω.

The main result of the paper [2] is the characterization of those weight functions for which the analogue of Whitney's extension theorem holds. Extending previous results of Meise and Taylor [6] and Bonet, Meise and Taylor [1] the following theorem is proved.

1. Theorem. *The following assertions are equivalent:*

(1) *There is* $C>1$ *such that for all* $y>0$: $\int_1^\infty \frac{\omega(yt)}{t^2}\,dt \le C\omega(y)+C.$

(2) *For each closed set* A *in* $\mathbb{R}^N$ *and each Whitney jet* F *of type* $\mathcal{E}_{\{\omega\}}$ *on* A *there is* $f\in\mathcal{E}_{\{\omega\}}(\mathbb{R}^N)$ *such that* F *is the restriction of* f *to* A.

(3) *For each closed set* A *in* $\mathbb{R}^N$ *and each Whitney jet* F *of type* $\mathcal{E}_{(\omega)}$ *on* A *there is* $f\in\mathcal{E}_{(\omega)}(\mathbb{R}^N)$ *such that* F *is the restriction of* f *to* A.

For the precise definition of Whitney jets we refer to our article [2].

The basic idea of the proof of the above result in the case of $\mathcal{E}_{\{\omega\}}$ goes back to Bruna [4], who indicated that the analogue of Whitney's extension theorem holds in a class of non-quasianalytic functions if it holds for the point and if the class contains cut-off functions satisfying certain estimates. Since it had already been shown that Whitney's extension theorem holds for the point if and only if the weight ω satisfies condition (1) in the theorem, the main step in the proof is to construct special cut-off functions when ω satisfies condition (1). This is done using Hörmander $\bar{\partial}$-method.

To be more precise, for a weight function ω its Young conjugate is defined by $\omega^*(s) := \sup\{\omega(t)-ts:\ t\ge 0\}$, $s>0$.

2. Lemma. *Let* ω *be a weight function satisfying condition* (1) *and which is concave and satisfies* $\omega(0)=0$. *Then for each* $\nu\in\mathbb{N}$ *there are* $m\in\mathbb{N}$, $M>0$ *and* $0<r_o<2^{-1}$ *such that for each* $0<r<r_o$ *there is* $g_{\nu,r}\in C^\infty(\mathbb{R})$ *which has the following properties:*

(i) $0\le g_{\nu,r}\le 1$, $\operatorname{supp}(g_{\nu,r}) \subset [-\frac{9r}{8}, \frac{9r}{8}]$ *and* $g_{\nu,r}|[-r,r] \equiv 1$

(ii) $\sup_{x\in\mathbb{R}} \sup_{j\in\mathbb{N}_o} |g_{\nu,r}^{(j)}(x)| \exp(-\frac{1}{m}\varphi^*(mj)) \le M \exp(\frac{1}{\nu}\omega^*(\nu r)).$

The results presented here were used by Kaballo [5] to derive estimates for the distribution of the eigenvalues of integral operators with ultradifferentiable kernels.

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