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On Certain Subsets of Bochner Integrable Functions Spaces

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INTRODUCION AND NOTATIONS

One of the most important methods used in the literature to introduce new properties in a Banach space E, consists in establishing some non trivial relationships between different classes of subsets of E. For instance, E is reflexive, or has finite dimension, if and only if every bounded subset is weakly relatively compact or norm relatively compact, respectively.

On the other hand, Banach spaces of the type $C(K)$ and $L_p(\mu)$ play a vital role in the general theory of Banach spaces. Their structure is so rich that many important concepts and results of the general theory have been modelled on these spaces. Also, the characterization of most important classes of subsets of these spaces, is well known. However, the situation is completely different for the analogous spaces of vector valued functions. In general, their structure is quite more involved than that of the scalar function spaces.

In this talk we shall be mainly concerned with the space $L_1(\mu,E)$. When $E = \mathbb{K}$, most of the classes of subsets we are interested in, coincide. This is no longer true in the vectorial case, and we shall try to determine classes of Banach spaces E for which the natural extension of the characterizations of several classes of distinguished subsets of $L_1(\mu)$, are valid in $L_1(\mu,E)$.

For the sake of convenience, we deal with real Banach spaces. We shall use the standar terminology in Banach spaces theory, as in [10] and [18]. If E is a Banach space, $B(E)$ will be its closed unit ball and E^* its topological dual. The word operator will always mean linear bounded operator. A series Σx_n in E is said to be weakly unconditionally Cauchy (*w.u.c.* in short) if $\Sigma\, |x^*(x_n)| < \infty$ for every $x^* \in E^*$ (equivalently, if $\{\Sigma_\sigma x_n : \sigma \subset \mathbb{N} \text{ finite}\}$ is a bounded subset). Throughout the paper, (Ω,Σ,μ) will be a finite measure space and for every p, $1 \leqslant p \leqslant \infty$, $L_p(\mu,E)$ will denote the usual Banach space of all (equivalence classes of) strongly E-valued measurable functions f on Ω, such that

$$\|f\|_p = \left[\int \|f(\omega)\|^p \, d\mu(\omega)\right]^{1/p} < \infty \quad (\text{if } 1 \leqslant p < \infty)$$

or

$$\|f\|_\infty = \text{ess sup}\,\{\, \|f(\omega)\| : \omega \in \Omega\} < \infty .$$

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I. Some Classes of Subsets of a Banach Space

As we mentioned at the introduction, one of the main methods to introduce new Banach spaces properties, follows the following general scheme: let $\mathscr{H}$ and $\mathscr{G}$ be classes of subsets of Banach spaces (so that, $\mathscr{H}(E)$ and $\mathscr{G}(E)$ are classes of subsets of E, for every Banach space E). Then we say that E has property $(\mathscr{H},\mathscr{G})$ if $\mathscr{H}(E)\subseteq\mathscr{G}(E)$. Let us denote by $\mathscr{B}$, $\mathscr{W}$, $\mathscr{WC}$ and $\mathscr{K}$ the classes of bounded, weakly relatively compact, weakly conditionally compact (i.e., $A\in\mathscr{WC}(E)$ if every sequence in A has a weakly Cauchy subsequence) and norm relatively compact subsets. Then we get that finite dimensionality is just property $(\mathscr{B},\mathscr{K})$, and reflexivity is property $(\mathscr{B},\mathscr{W})$, whereas Rosenthal's ℓ_1-theorem (see [10, Ch. XI]) establishes that a Banach space has property $(\mathscr{B},\mathscr{WC})$ if and only if it contains no copy of ℓ_1.

Now we shall define two other classes of subsets of a Banach space E:

Definition I.1. A subset A of E is called Dunford–Pettis set (resp., a (V^*) set) if for every weakly null sequence (x_n^*) (resp., for every *w.u.c.* series Σx_n^*, see the introduction) in E^*, the following holds:

$$\lim_{n\to\infty}\sup\{|x_n^*(x)|:x\in A\}=0.$$

(V^*) sets were introduced by Pelczynski in [19], whereas Dunford–Pettis set were defined by Andrews in [1]. Let us denote by $\mathscr{DP}(E)$ and $\mathscr{V}^*(E)$ the families of Dunford–Pettis and (V^*)–sets in E. Next lemma is an easy and useful characterization of these classes in terms of operators:

Lemma I.2. *Let A be a subset of a Banach space E.*
a) ([1]) *$A\in\mathscr{V}^*(E)$ if and only if $T(A)$ is relatively compact, for every operator T from E into ℓ_1.*
b) ([12]) *$A\in\mathscr{DP}(E)$ if and only if $T(A)$ is relatively compact, for every operator T from E into c_0.*

Then, the following relationships hold:

$$\mathscr{K}\subseteq\mathscr{W}\subseteq\mathscr{WC}\subseteq\mathscr{V}^*\subseteq\mathscr{B} \tag{1}$$

and

$$\mathscr{K}\subseteq\mathscr{DP}\subseteq\mathscr{WC},$$

and the inclusions are, in general, strict.

If we now apply the general scheme to the classes we have introduced, besides the properties we have already mentioned, we get:

– Property $(\mathscr{W},\mathscr{DP})$ coincides with property $(\mathscr{WC},\mathscr{DP})$, and is called Dunford–Pettis property. It was introduced by Grothendieck in [15], and has been intensively studied.

– Property $(\mathscr{W},\mathscr{K})$ coincides with property $(\mathscr{WC},\mathscr{K})$ and is the well known Schur property.

– Property $(\mathcal{WC}, \mathcal{W})$ is simply the weak sequential completeness.

– Property $(\mathcal{V}^*, \mathcal{W})$ is called (V^*) property. It was introduced by Pelczynski in [19]. From (1) it is obvious that (V^*) property is equivalent to the weak sequential completeness (property $(\mathcal{WC}, \mathcal{W})$) and property $(\mathcal{V}^*, \mathcal{WC})$. This last property was introduced in [3], with the name of property weak (V^*).

– Property $(\mathcal{B}, \mathcal{V}^*)$ is equivalent to the non containment of complemented copies of ℓ_1 [3, Cor. 1.5].

– Property $(\mathcal{B}, \mathcal{DP})$ expresses precisely the fact that the dual of the Banach space under consideration, has the Schur property [4, Prop. II.10].

– Property $(\mathcal{DP}, \mathcal{W})$ was called by Leavelle, who introduced it in [17], *RDP** property.

– Finally, property $(\mathcal{DP}, \mathcal{K})$ has been recently studied by Emmanuele in [13].

The general scheme that is under all these properties, allows to prove at once several common facts shared by all of them. See [4] for more details.

Now, we shall collect some general properties of the classes of subsets that we have introduced. Most proofs follow easily from definitions and the general theory (see [4]):

LEMMA I.3. *Let $\mathcal{H}$ be any of the classes $\mathcal{K}$, $\mathcal{W}$, $\mathcal{WC}$, $\mathcal{DP}$ or $\mathcal{V}^*$.*

a) $\mathcal{H}$ is preserved by continuous linear images, linear combinations, closed absolutely convex hulls, finite products and passing to subsets.

b) A belongs to $\mathcal{H}$ if and only if every countable subset of A belongs to $\mathcal{H}$.

c) If A is a subset of the Banach space E and for each $\epsilon > 0$ there is an $A_\epsilon \in \mathcal{H}(E)$ such that $A \subseteq A_\epsilon + \epsilon B(E)$, then $A \in \mathcal{H}(E)$.

II. DISTINGUISHED SUBSETS OF $L_1(\mu,E)$

The space $L_1(\mu)$ has the Dunford–Pettis property [15] and the $(\mathcal{V}^*)$ property [19]. Hence, in this case, the classes $\mathcal{V}^*$, $\mathcal{WC}$, $\mathcal{DP}$ and $\mathcal{W}$ coincide. By the well known Dunford–Pettis criteria (see, f.i., [10, Ch. VII]), they are precisely the bounded and uniformly integrable subsets. This is no longer true in the vectorial case $L_1(\mu,E)$ and, in fact, there are no complete characterization of any of the aforementioned classes. This has been one of the main difficulties to solve the long standing open questions of when $L_1(\mu,E)$ inherits "good" properties from $L_1(\mu)$ and E. As far as we know, the only complete satisfactory answer was given by Talagrand in [23], proving that $L_1(\mu,E)$ is weakly sequentially complete if and only if so is E. Previously, Talagrand has also proved in [22] the existence of a Banach space with the Dunford–Pettis property E (even a Schur space) such that $L_1(\mu,E)$ does not have the Dunford–Pettis property, where μ is the Lebesgue measure on [0,1]. Some partial answers about when $L_1(\mu,E)$ inherits the weak (V^*) or the (V^*) property from E, can be found in [3] and [21].

Next results shows some necessary conditions for a set in $L_1(\mu,E)$ to belong to any of the classes we are interested in.

PROPOSITION II.1. *Let $\mathscr{H}$ be any of the classes $\mathscr{K}$, $\mathscr{W}$, $\mathscr{WC}$, $\mathscr{DP}$ or $\mathscr{V}^*$ and E a Banach space. If $K \in \mathscr{H}(L_1(\mu,E))$, then*

a) K is bounded.

b) K is uniformly integrable, i.e.,

$$\lim \sup_{\mu(A)\to 0} \left\{ \int_A \|f\| \, d\mu : f \in K \right\} = 0$$

c) For every $A \in \Sigma$,

$$K(A) = \left\{ \int_A f \, d\mu : f \in K \right\} \in \mathscr{H}(E).$$

This is a consequence of Proposition 3.1 of [3], where it is proved that every (V^*) set in $L_1(\mu,E)$ is uniformly integrable, and the fact that, for every $A \in \Sigma$, the map

$$f \in L_1(\mu,E) \longmapsto \int_A f \, d\mu \in E$$

is linear continuous.

From now on, $\mathscr{H}$ will always have the same meaning as in the above Proposition. Conditions *a)* to *c)* are then the natural extension to the vectorial setting of the characterization of sets in $\mathscr{H}(L_1(\mu))$. But they are by no means sufficient to guarantee that a subset K belongs to $\mathscr{H}(L_1(\mu,E))$, as the following example shows:

EXAMPLE II.2. Let μ be the Lebesgue measure on $\Omega = [0,1]$, $E = \ell_1$, (e_n) the usual unit basis in E and (r_n) the sequence of Rademacher functions. Let us consider the set

$$K = \{ r_n e_n : n \in \mathbb{N} \} \subseteq L_1(\mu,E).$$

Clearly, $\| r_n e_n \|_1 = 1$ and $\int_A \| r_n e_n \| d\mu = \mu(A)$ for every $n \in \mathbb{N}$. Finally, for every $A \in \Sigma$,

$$\lim_{n\to\infty} \left\| \int_A r_n e_n \, d\mu \right\| = \lim_{n\to\infty} \left| \int_A r_n \, d\mu \right| = 0.$$

In particular, $K(A) \in \mathscr{K}(E)$. Hence K satisfies conditions *a) − c)* of Proposition II.1. However, K is not even a (V^*)−set. In fact, if e_n^* denotes the nth unit vector in $\ell_\infty \approx \ell_1^*$, the sequence $\varphi_n = r_n e_n^*$ belongs to $L_\infty(\mu,\ell_\infty) \subseteq L_1(\mu,\ell_1)^*$, and $\Sigma \varphi_n$ is *w.u.c.*, because for every finite subset σ of $\mathbb{N}$,

$$\| \textstyle\sum_{n\in\sigma} \varphi_n \|_\infty = 1$$

However, $\langle r_n e_n, \varphi_n \rangle = 1$ for every n. ∎

For the sake of brevity, we shall give the following

DEFINITION II.3.

a) A subset K of $L_1(\mu,E)$ satisfying conditions *a)* to *c)* of Proposition II.1, will be called a $\mu\mathscr{H}$−set.

b) A Banach space E is said to have property $P(\mu,\mathscr{H})$ if every $\mu\mathscr{H}$−set belongs to $\mathscr{H}(L_1(\mu,E))$.

So a Banach space E has property $P(\mu,\mathscr{H})$ if the natural extension of the characterization of $\mathscr{H}$ − sets in $L_1(\mu)$ is valid for $L_1(\mu,E)$.

The next proposition follows easily from the definitions and lemma I.3.

PROPOSITION II.4.

a) The family $\mu\mathscr{H}(E)$ is stable under linear combinations, taking closed absolutely convex hulls and passing to subsets.

b) $K \subseteq L_1(\mu,E)$ is a $\mu\mathscr{H}$ − set if and only if every countable subset of K is a $\mu\mathscr{H}$ − set.

c) If F is another Banach space and $T \in \mathscr{L}(E,F)$, for every $\mu\mathscr{H}$ − set $K \subseteq L_1(\mu,E)$, the set $\{f \circ T : f \in K\} \subseteq L_1(\mu,F)$ is also a $\mu\mathscr{H}$ − set.

d) If $K \subseteq L_1(\mu,E)$ and for every $\epsilon > 0$ a $\mu\mathscr{H}$ − set K_ϵ exists, so that $K \subseteq K_\epsilon + \epsilon B(L_1(\mu,E))$, then K is also a $\mu\mathscr{H}$ − set.

e) Let $\mathscr{G}$ be another of the classes considered in Proposition II.1. If E has property $(\mathscr{H},\mathscr{G})$ (see section I) and property $P(\mu,\mathscr{G})$ then $L_1(\mu,E)$ has property $(\mathscr{H},\mathscr{G})$.

Part *e)* above shows the interest in knowing when a Banach space has property $P(\mu,\mathscr{H})$. When μ is a purely atomic measure, the answer is very easy. In fact, the following result holds:

PROPOSITION II.5. *If μ is purely atomic, every Banach space has property $P(\mu,\mathscr{H})$.*

The proof is based on the fact that, in this case, μ is concentrated on a countable set (A_n) of atoms, and every $f \in L_1(\mu,E)$ is constant on each A_n. Hence, $L_1(\mu,E)$ is isometric to $\ell_1(E)$, and the uniform integrability of a set $K \subseteq L_1(\mu,E)$ means that the corresponding subset $\overline{K}$ of $\ell_1(E)$ has "uniformly small tails", i.e.

$$\overline{K} \subseteq P_n(\overline{K}) + \epsilon_n B(\ell_1(E)) \text{ , for every } n \in \mathbb{N}$$

where P_n is the canonical projection $P_n(x) = (x_1,...,x_n,0,0...)$ and (ϵ_n) decreases to 0. Condition *c)* of the definition of $\mu\mathscr{H}$ − set yields that $P_n(\overline{K}) \in \mathscr{H}(\ell_1(E))$ for every n, and lemma I.3 applies.

An immediate consequence is the following:

COROLLARY II.6. *Let $\mathscr{H}$ and $\mathscr{G}$ be any of the classes we have considered in Proposition II.1. For a Banach space E, the following assertions are equivalent:*

a) E has property $(\mathscr{H},\mathscr{G})$.

b) $\ell_1(E)$ has property $(\mathscr{H},\mathscr{G})$.

All the above results are probably known. But also the proofs are probably different for each property arising from a concrete choice of $\mathscr{H}$ and $\mathscr{G}$.

The only known complete characterization of a property $P(\mu,\mathscr{H})$ is, as far as we know, the corresponding to the case $\mathscr{H} = \mathscr{W}$:

THEOREM II.7. ([14]) *A Banach space E has property $P(\mu,\mathscr{W})$ if and only if both E and E^* have the Radon−Nikodym property with respect to μ.*

When μ is the Lebesgue measure on [0,1], the above result was proved independently by Ghoussoub and Saab in [16].

The next result is a characterization of the property $P(\mu, \mathscr{V}^*)$.

THEOREM II.8. *Let μ be a non purely atomic measure. For a Banach space E, the following assertions are equivalent:*

i) E contains no complemented copy of ℓ_1.

ii) Every bounded and uniformly integrable subset of $L_1(\mu,E)$ is a $(V^*)-$set.

iii) E has property $P(\mu, \mathscr{V}^*)$.

That *ii)* implies *iii)* is obvious, and the proof of *iii)* $\Longrightarrow$ *i)* follows the lines of example II.2. In fact, let (x_n) be a normalized sequence equivalent to the unit basis of ℓ_1, spanning a complemented subspace, and let (x_n^*) be the associated functionals. If (r_n) is a sequence of "Rademacher-like" functions on Ω (which can be built because μ is not purely atomic; see [5, Th. 9]), the set

$$K = \{ r_n x_n : n \in \mathbb{N}\} \subseteq L_1(\mu,E)$$

is a $\mu\, \mathscr{V}^*-$set, but not a $(V^*)-$set.

The proof that *i)* $\Longrightarrow$ *ii)* is more involved: Suppose E contains no complemented copy of ℓ_1 and let $K \subseteq L_1(\mu,E)$ be bounded and uniformly integrable. By lemma I.2, it suffices to prove that $T(\{f_n : n \in \mathbb{N}\})$ is relatively compact, for every sequence (f_n) in K and every operator T from $L_1(\mu,E)$ into ℓ_1. This is proved by making a careful study of a suitable representation of T.

As an easy consequence, we get the following:

COROLLARY II.9. *If $L_1(\mu,E)$ contains an uniformly integrable sequence (f_n) equivalent to the unit basis of ℓ_1 and spanning a complemented subspace, then E has a complemented copy of ℓ_1.*

Property $P(\mu, \mathscr{W}\mathscr{C})$ can be characterized as follows:

THEOREM II.10. *Let μ be a non purely atomic measure. For a Banch space E, the following assertions are equivalent:*

i) E contains no copy of ℓ_1.

ii) Every bounded and uniformly integrable subset of $L_1(\mu,E)$ is weakly conditionally compact.

iii) If $p > 1$, every bounded subset of $L_p(\mu,E)$ is weakly conditionally compact.

iv) E has property $P(\mu, \mathscr{W}\mathscr{C})$.

The statement *i)* $\Longrightarrow$ *ii)* is precisely corollary 9 of [8]. Obviously, *ii)* implies *iv)*, and *ii)* implies *iii)* because corollary 4 of [6] asserts that a bounded subset of $L_p(\mu,E)$ $(1 < p < \infty)$ is weakly (conditionally) compact if and only if it is weakly (conditionally) compact in $L_1(\mu,E)$ (in other words, the natural inclusion of $L_p(\mu,E)$ into $L_1(\mu,E)$ is a tauberian operator).

The proofs of *iii)* $\Rightarrow$ *i)* and *iv)* $\Rightarrow$ *i)* are very similar: If $(x_n) \subseteq E$ is a sequence equivalent to the unit vector basis of ℓ_1, and (r_n) are Rademacher–like functions, as in the proof of Theorem II.7, then $(r_n \otimes x_n)$ is equivalent to the unit basis of ℓ_1 in any space $L_p(\mu,E)$ $(1 \leqslant p < \infty)$, but $K = \{r_n \otimes x_n : n \in \mathbb{N}\}$ is even a $\mu\mathscr{K}$–set.

The next result follows at once:

COROLLARY II.11.

a) If $L_1(\mu,E)$ contains a uniformly integrable sequence equivalent to the unit basis of ℓ_1, then E contains a copy of ℓ_1.

b) ([20]) *If E contains no copy of ℓ_1, then $L_p(\mu,E)$ does not contain it either, for $1 < p < \infty$.*

As for property $P(\mu, \mathscr{DP})$, we have only the following partial answer:

THEOREM II.12. *Let E be a Banach space.*

a) If E^* has the Schur property, then for every μ, any bounded and uniformly subset of $L_1(\mu,E)$ is a Dunford–Pettis set. In particular, E has property $P(\mu, \mathscr{DP})$, for every μ.

b) If E has property $P(\mu, \mathscr{DP})$ for some non purely atomic measure μ, then E contains no copy of ℓ_1. In particular, if E has the Dunford–Pettis property, then it has property $P(\mu,E)$ if and only if E^* is a Schur space.

The proof is similar to that of Theorem II.8.

Part *a)* of the above theorem was proved in [1, Cor. 4], with a different argument. It follows immediately that $L_1(\mu,E)$ has the Dunford–Pettis property when E^* is Schur. However, as we mentioned before, Talagrand built in [22] a Schur space E such that $L_1(\lambda,E)$ does not have the Dunford–Pettis property (λ = Lebesgue measure on $[0,1]$). On the other hand, Bourgain proved in [9] that $L_1(\mu,C(K))$ and all its duals have the Dunford–Pettis property.

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