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Polynomial Characterizations of the Dunford–Pettis Property

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We introduce and characterize the class $\mathscr{P}_{wd}$ of polynomials between Banach spaces whose restrictions to Dunford – Pettis (DP) sets are weakly continuous. All the weakly compact and the scalar valued polynomials belong to $\mathscr{P}_{wd}$. We prove that a Banach space E has the Dunford – Pettis (DP) property if and only if every $P \in \mathscr{P}_{wd}$ is weakly sequentially continuous. This result contains a characterization of the DP property given in [3], answering a question of Pelczyńsky: E has the DP property if and only if any weakly compact polynomial on E takes weak Cauchy sequences into convergent ones. It also extends other characterizations of the DP property by operators to the case of polynomials. Other properties of polynomials between Banach spaces are obtained.

E, F and G will denote Banach spaces. E^* will be the dual of E, and $\mathbb{N}$ the set of natural numbers. A subset $A \subset E$ is a DP set if for any weakly compact operator $T: E \longrightarrow F$, $T(A)$ is relatively compact in F. We say that $A \subset E$ is a Rosenthal subset if every sequence in A has a weak Cauchy subsequence. An operator $T: E \longrightarrow F$ is completely continuous if it takes weakly null sequences into null sequences. T is Rosenthal if it takes the unit ball of E into a Rosenthal subset of F. E has the DP property if every weakly compact operator $T: E \longrightarrow F$ is completely continuous.

For $n \in \mathbb{N}$, $L(^nE,F)$ denotes the vector space of all n – linear continuous mappings from $E^n = E \times \overset{(n)}{\cdots} \times E$ into F. Any map $P: E \longrightarrow F$ of the form $P(x) = A(x,...,x)$, where $A \in L(^nE,F)$, is called an n – homogeneous continuous polynomial, and we shall write $P \in \mathscr{P}(^nE,F)$. When F is the scalar field, we simply write $L(^nE)$ or $\mathscr{P}(^nE)$, respectively. $P \in \mathscr{P}(^nE,F)$ is said to be (weakly) compact if it takes the unit ball of E into a relatively (weakly) compact subset of F.

We shall need the following characterizations of the ideal $WCo^{-1} \circ Co$ of all operators $T: E \longrightarrow F$ such that $K \circ T$ is compact for every weakly compact operator $K: F \longrightarrow G$:

PROPOSITION 1. *Let $T: E \longrightarrow F$ be an operator and $T^*: F^* \longrightarrow E^*$ its adjoint. The following assertions are equivalent:*
a) T belongs to the operator ideal $WCo^{-1} \circ Co$.

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b) For every weakly compact operator $K: F \longrightarrow c_0$, $K \circ T$ *is compact.*

c) TB_E *is a* DP *set in* F.

d) T^* *is completely continuous.*

We denote by $C_{wk}(E,F)$ and $C_{wd}(E,F)$ the space of all mappings $f: E \longrightarrow F$ which are weakly continuous when restricted to weakly compact and DP sets, respectively. $C_{wk}(E,F)$ can also be described as the space of all weakly sequentially continuous mappings, i.e. mappings taking weakly convergent sequences into convergent ones. It is easy to show that $C_{wk}(E,F) \subseteq C_{wd}(E,F) \subseteq C(E,F)$, where $C(E,F)$ is the space of all continuous mappings from E to F. We denote $\mathcal{P}_{w\alpha}(^nE,F) = C_{w\alpha}(E,F) \cap \mathcal{P}(^nE,F)$ for $\alpha = k$ or d.

For an operator $S: G \longrightarrow E$, we shall denote by $S^k: G^k \longrightarrow E^k$ the operator given by $S^k(x_1,...,x_k) = (Sx_1,...,Sx_k)$, where $x_1,...,x_k \in G$.

THEOREM 2. *Let* $S: G \longrightarrow E$ *be an operator in the ideal* $WCo^{-1} \circ Co$. *Then, for every* $k \in \mathbb{N}$, *the operator* $S^*_k: L(^kE) \longrightarrow L(^kG)$ *given by* $S^*_k(A) = A \circ S^k$, *is completely continuous.*

COROLLARY 3. ([4]) *If* E^* *has the Schur property, then so have* $L(^kE)$ *and* $\mathcal{P}(^kE)$ *for every* $k \in \mathbb{N}$.

THEOREM 4. *The spaces* $\mathcal{P}(^kE)$ *and* $\mathcal{P}_{wd}(^kE)$ *coincide for every* $k \in \mathbb{N}$.

PROPOSITION 5. *Given* $k \in \mathbb{N}$ *and a polynomial* $P \in \mathcal{P}(^kE,F)$, *the following assertions are equivalent:*

a) $P \in \mathcal{P}_{wd}(^kE,F)$.

b) For any operator $S: G \longrightarrow E$ *in the ideal* $WCo^{-1} \circ Co$, $P \circ S$ *is compact.*

c) P *takes* DP *sets into relatively compact ones.*

d) P *is weakly uniformly continuous on* DP *sets of* E.

Given a polynomial $P \in \mathcal{P}(^kE,F)$, its adjoint is the bounded linear map $P^*: F^* \longrightarrow \mathcal{P}(^kE)$, defined by $P^*(\psi) = \psi \circ P$ ($\psi \in F^*$). It is proved in [5, Prop. 2.1] that $P \in \mathcal{P}(^kE,F)$ is weakly compact if and only if P^* is weakly compact.

PROPOSITION 6. *A polynomial* $P \in \mathcal{P}(^kE,F)$ *has Rosenthal adjoint if and only if there exists a space* G *whose dual contains no copy of* ℓ_1, *a polynomial* $\tilde{P} \in \mathcal{P}(^kE,G)$ *and an operator* $j: G \longrightarrow F$ *such that* $j \circ \tilde{P} = P$.

THEOREM 7. *If* $P \in \mathcal{P}(^kE,F)$ *has Rosenthal adjoint, then* $P \in \mathcal{P}_{wd}(^kE,F)$.

THEOREM 8. *The following assertions are equivalent:*

a) E *has the Dunford–Pettis property.*

b) For every Banach space F *and* $n \in \mathbb{N}$, $\mathcal{P}_{wd}(^nE,F) = \mathcal{P}_{wk}(^nE,F)$.

c) For every Banach space F *and* $n \in \mathbb{N}$, *every polynomial* $P \in \mathcal{P}(^nE,F)$ *with Rosenthal adjoint belongs to* $\mathcal{P}_{wk}(^nE,F)$.

d) For every Banach space F *and* $n \in \mathbb{N}$, *every weakly compact polynomial* $P \in \mathcal{P}(^nE,F)$ *belongs to* $\mathcal{P}_{wk}(^nE,F)$.

e) For some $n \in \mathbb{N}$, *every weakly compact polynomial* $P \in \mathcal{P}(^nE, c_0)$ *belongs to* $\mathcal{P}_{wk}(^nE, c_0)$.

The equivalence a) $\Leftrightarrow$ *b)* was obtained by Ryan [3], answering a question of Pelczyński [2]. The relations *a)* $\Leftrightarrow$ *e)* and *a)* $\Leftrightarrow$ *c)* for $n = 1$ may be found in [2, Prop. 4] and [1, Th. 11], respectively.

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