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Extending Automorphisms to the Rational Fractions Field

FERNANDO FERNÁNDEZ RODRÍGUEZ AND AGUSTÍN LLERENA ACHUTEGUI

Dpto. de Economía Aplicada, Univ. Las Palmas, Las Palmas de Gran Canaria, Spain

Dpto. de Matemáticas, Univ. Alcalá de Henares, Madrid, Spain

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We say that a field $\mathbb{K}$ has the Extension Property if every automorphism of $\mathbb{K}(X)$ extends to an automorphism of $\mathbb{K}$. J.M. Gamboa and T. Recio [2] have introduced this concept, naive in appearance, because of its crucial role in the study of homogeneity conditions in spaces of orderings of functions fields. Gamboa [1] has studied several classes of fields with this property: Algebraic extensions of the field $\mathbb{Q}$ of rational numbers; euclidean, algebraically closed and pythagorean fields; fields with an unique archimedean ordering. We have introduced an apparently stronger type of extension property, simplifying some techniques and broadening the results.

DEFINITION. $\mathbb{K}$ has the Generalized Extension Property (in short: $\mathbb{K}$ is G.E.P.) if for every field L and every field homomorphism $\varphi \colon \mathbb{K} \longrightarrow L(T)$ verifies that $\varphi(\mathbb{K}) \subseteq L$.

With similar techniques to Gamboa J.M. [1], we can improve the above mentioned results: Every prime field is G.E.P. Every algebraic extension of a G.E.P. field is G.E.P., (in particular, finite ones). Euclidean, pythagorean and algebraically closed fields are G.E.P.

Our main result is the following:

THEOREM. *Let* $(\mathbb{K},|\ |)$ *be a complete valued field. Then* $\mathbb{K}$ *has the Generalized Extension Property.*

We give at the bottom a brief outline of the proof:

If K is archimedian, then it is isomorphic to the reals field $\mathbb{R}$ or to the complexes field $\mathbb{C}$ via Ostrowski's theorem (Jacobson [3]).

For K non archimedian we have to establish a Lemma.

Let K a field and let $a \in K$ such that equation $x^n - a = 0$, has roots in K for infinitely many values of $n \in \mathbb{N}$. Then, given the homomorphism $\varphi \colon K \longrightarrow L(T)$, it is true that $\varphi(a) \in L$.

It follows from the fact that, for every $a \in K$, supposing that $\varphi(a) = f_a / g_a$ (f_a, g_a coprime polynomials) doesn't belong to L, we will get an absurd if we consider a root α of a with order larger than $d(a) = \deg(f_a) + \deg(g_a) \in \mathbb{Z}$.

Given $s \in A$ not unity (A is the valuation ring of K), the proof of the main theorem follows, via Hensel's Lemma (Lang [4]), finding an element $a \in A$ that verifies $a^r = 1 + s$ for infinitely many values of r. Then $\varphi(1+s) \in L$ and it is easy to proof that $\varphi(s) \in L$.

A corollary of the above Theorem is that purely transcendental extension fields of a field are not G.E.P. Thus, complete valued fields (e.g., p−adic ones, series fields,...) are not purely transcendental extensions of some other field.

We have also studied the subfields that are G.E.P. The proofs do not present special difficulties:

PROPOSITION. *Let K be a field and let $(K_i)_{i\in I}$ be a family of* G.E.P. *subfields. Let K_I be the subfield generated by $\bigcup_{i\in I} K_i$. Then K_I is* G.E.P.

From the above proposition and from the fact that prime fields are G.E.P., one has:

COROLLARY. *Let K be a field. Then there exists a maximun* G.E.P. *subfield in K, which we shall denote by $Gep(K)$.*

There is also the following:

COROLLARY. *Let K be a field and $M \subset K$ a subfield. Then:*

i) $Gep(M) \subset Gep(K)$.

ii) *K is* G.E.P. *if and only if* $Gep(K) = K$.

iii) $Gep(K(T_1,\dots,T_n)) = Gep(K)$.

iv) $Gep(K(T_1,\dots,T_n)) = K$ *if and only if K is* G.E.P.

v) *There is no* G.E.P. *subfield of* $K(T_1,\dots,T_n)$ *containing (strictly) K.*

vi) $Gep(K)$ *is algebraically closed in K.*

After the last Corollary, it is natural to consider whether the only fields that are not G.E.P. are those which are purely transcendental extensions of some other field. The answer is negative:

PROPOSITION. *There exists fields that are neither* G.E.P. *nor purely transcendental extensions of other fields.*

If every non G.E.P. field were a purely transcendental extension of other field, after *v)* in the last Corollary we could find that, in particular, every subfield of $\mathbb{C}(X_1,\dots,X_n)$ containing $\mathbb{C}$ should be of type $\mathbb{C}(T_1,\dots,T_r)$, where $X_1,\dots,X_n$ and $T_1,\dots,T_r$ are sets of algebraically independent variables.

This is the conjecture known as "Lüroth's problem", which is, nevertheless, false: In Shafarevich [5], one can see that the rational functions field L of the algebraic manifold

$$V = \{(x,y,z,t) \in \mathbb{C}^4 : 1 + x^3 + y^3 + z^3 + t^3\} = 0$$

is a subfield contradicting yhe above conjecture, for V is an unirrational but not a birrational manifold. This ends our proof.

Finally we want to introduce some open problems. Within the frame of the question whether every Extension Property field is also a G.E.P. field, we can ask the following: Which G.E.P. field properties are valid for E.P. fields? We know that for any given field there is an algebraic extension thereof which is G.E.P. Does it exist a finite extension which is G.E.P.? Does it exist a minimal algebraic extension which is G.E.P.? We formulate again a non-solved problem for E.P. fields: If K has a unique ordering, is K G.E.P.?

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