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Monographic Study of a Cubic Field

FRANCISCA CÁNOVAS ORVAY

E.T.S. de Ingenieros de Caminos, Canales y Puertos, Univ. Politécnica, Madrid, Spain

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Let K be a cubic field. We can suppose that $K = \mathbb{Q}(\theta)$, where $Irr(\theta,\mathbb{Q}) = x^3 - ax + b$, with $a,b \in \mathbb{Z}$ and we can assume that $p^2 \mid a$, $p^3 \mid b$ is not satisfied for any prime p. We have $\text{disc}(\theta) = 4a^3 - 27b^2$ and we put $4a^3 - 27b^2 = d^2q$ where q is free of squares. We denote by D the discriminant of K. We can see that $\theta_1 = (4a^2 - 9b\theta - 6a\theta^2) / d$ is an algebraic integer.

For every prime $p \in \mathbb{Z}$ and integer $m \in \mathbb{Z}$ we denote by $v_p(m)$ the greatest exponent r such that $p^r \mid m$. We say that a $\mathbb{Q}$−basis of integers $\{\alpha_1,\alpha_2,\alpha_3\}$ is minimal in p if $v_p(\text{disc}(\alpha_1,\alpha_2,\alpha_3)) = v_p(D)$.

We apply theorem 9.28 in [1] to obtain a minimal basis in each prime of $\mathbb{Z}$ in terms of a defining polynomial of K. We apply this theorem to $\{1,\theta,\theta^2\}$ and $\{1,\theta^2,\theta_1\}$; with both of them we have covered all the posibles cases.

As a consequence, we obtain the discriminant D of K, which has been obtained by P. Llorente and E. Nart [4], using another methods. Our study has also application to obtain integral basis for K cubic field.

It is known (see, for example, [2] and [3]) that the discriminant of a cyclic cubic field is p^2 where $p = 3^\delta p_1 \cdots p_r$, $\delta \in \{0,2\}$, $p_i \equiv 1(3)$ pairwise different primes. Using our previous results and the law of quadratic reciprocity we give a new and completely elementary proof of this result about the discriminant of a cyclic cubic field.

We are going to schematize our results.

<u>MINIMAL BASIS IN 2 FOR *K* CUBIC FIELD</u>

$a \equiv$	$b \equiv$	Another Conditions	Minimal Basis in 2	$v_2(D)$
0(4)	0(4)	$b \not\equiv 0(8)$	$\{1,\theta,\theta^2/2\}$ $\{1,\theta^2/2,\theta_1\}$	2
0(2)	0(2)	$b \not\equiv 0(4)$	$\{1,\theta^2,\theta_1\}$ $\{1,\theta,\theta^2\}$	2
0(2)	0(8)	$a \not\equiv 0(4)$	$\{1,\theta,\theta^2/2\}$	3
0(2)	0(4)	$a \not\equiv 0(4)$ $b \not\equiv 0(8)$	$\{1,\theta,\theta^2/2\}$ $\{1,\theta^2/2,\theta_1\}$	2

$a \equiv$	$b \equiv$	Another Conditions	Minimal Basis in 2	$v_2(D)$
	1(2)		$\{1,\theta,\theta^2\}$ $\{1,\theta^2,\theta_1\}$	0
1(2)	0(2)	$b \not\equiv 0(4)$ $q \equiv 1(4)$	$\{1,\theta^2,(\theta^2+\theta_1)/2\}$	0
1(2)	0(2)	$b \not\equiv 0(4)$ $q \equiv 3(4)$	$\{1,\theta^2,\theta_1\}$	2
1(2)	0(2)	$b \not\equiv 0(4)$ $q \equiv 0(2)$	$\{1,\theta^2,\theta_1\}$	3
1(4)	0(4)		$\{1,\theta,(\theta+\theta^2)/2\}$	0
3(4)	0(8)		$\{1,\theta,\theta^2\}$	2
3(4)	0(4)	$b \not\equiv 0(8)$	$\{1,\theta,\theta^2\}$ $\{1,\theta^2,(\theta_1+\theta^2)/2\}$	2

MINIMAL BASIS IN 3 FOR *K* CUBIC FIELD

$a \equiv$	$b \equiv$	Another Conditions	Minimal Basis in 3	$v_3(D)$
		$a \not\equiv 0(3)$	$\{1,\theta,\theta^2\}$	0
0(3)		$b^2 \not\equiv a+1(9)$ $a \not\equiv 3(9)$ $b \not\equiv 0(3)$	$\{1,\theta^2,\theta_1/3\}$ $\{1,\theta,\theta^2\}$	3
0(3)	0(3)	$a,b \not\equiv 0(9)$	$\{1,\theta,\theta^2\}$	3
0(3)	0(9)	$a \not\equiv 0(9)$	$\{1,\theta,\theta^2/3\}$	1
0(3)	1(3)	$b^2 \equiv a+1(9)$ $a \not\equiv 3(9)$	$\{1,\theta,(1-\theta+\theta^2)/3$	1
0(3)	2(3)	$b^2 \equiv a+1(9)$ $a \not\equiv 3(9)$	$\{1,\theta,(1+\theta+\theta^2)/3$	1
0(9)	0(9)	$b \not\equiv 0(27)$ $a \not\equiv 0(27)$	$\{1,\theta^2/3,\theta_1/3\}$ $\{1,\theta,\theta^2/3\}$	4

$a \equiv$	$b \equiv$	Another Conditions	Minimal Basis in 3	$v_3(D)$
0(9)	0(3)	$b \not\equiv 0(9)$	$\{1,\theta^2,\theta_1/3\}$ $\{1,\theta,\theta^2\}$	5
3(9)		$b^2 \equiv a+1(27)$ $q \equiv 0(3)$	$\{1,(-1+\theta^2)/3,\theta_1/3\}$	1
3(9)		$b^2 \equiv a+1(27)$ $q \not\equiv 0(3)$	$\{1,(-1+\theta^2)/3,\theta_1/3\}$	0
3(9)		$b^2 \equiv 4(9)$ $b^2 \not\equiv a+1(27)$	$\{1,\theta^2,\theta_1/3\}$	3
3(9)		$b^2 \not\equiv 4(9)$ $b \not\equiv 0(3)$	$\{1,\theta^2,\theta_1\}$ $\{1,\theta,\theta^2\}$	4
0(27)	0(9)	$b \not\equiv 0(27)$	$\{1,\theta^2/3,\theta_1/3\}$ $\{1,\theta,\theta^2/3\}$	5

MINIMAL BASIS IN $p > 3$ FOR K CUBIC FIELD

$a \equiv$	$b \equiv$	Another Conditions	Minimal Basis in p	$v_p(D)$
$0(p^2)$	$0(p^2)$	$b \not\equiv 0(p^3)$	$\{1,\theta,\theta^2/p\}$ $\{1,\theta^2/p,\theta_1\}$	2
$0(p)$	$0(p)$	$b \not\equiv 0(p^2)$	$\{1,\theta^2,\theta_1\}$ $\{1,\theta,\theta^2\}$	2
$0(p)$	$0(p^3)$	$a \not\equiv 0(p^2)$	$\{1,\theta,\theta^2/p\}$	1
$0(p)$	$0(p^2)$	$a \not\equiv 0(p^2)$ $b \not\equiv 0(p^3)$	$\{1,\theta,\theta^2/p\}$ $\{1,\theta_1/p,\theta^2/p\}$	1
		$b,q \not\equiv 0(p)$	$\{1,\theta^2,\theta_1\}$	0
		$b \not\equiv 0(p)$ $q \equiv 0(p)$	$\{1,\theta^2,\theta_1\}$	1
	$0(p)$	$a \not\equiv 0(p)$	$\{1,\theta,\theta^2\}$	0

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