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Baer Invariants of Crossed Modules

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Baer invariants of groups have been studied for long, but Fröhlich [5] was who, considering them in algebras, gave their first homological treatment. Their utility was shown in [8], [6] and [3], where they were considered in various Ω – group structures.

On the other hand, the notion of crossed module, though it arise naturally in topology [12], acquired relevancy when it played a decisive role in the interpretation of higher cohomology of groups. Since then, crossed modules were studied, as an algebraic object, in various structures. So was done for free and projective crossed modules in [10] and for coproducts in [1].

Since the category of (pre)crossed modules is not an algebraic category (on $\mathcal{Set}$), it seems interesting to ask if methods of [5] and [6] on Baer invariants can be applied to (pre)crossed modules. A particular instance of this question was treated in [4], where the "variety" of crossed modules, into the category of G – precrossed modules, was considered. Also, in [9] the actor and the centre of a crossed module were constructed. This fact suggests the application of the mentioned methods to the variety of abelian crossed modules.

The purpose of this paper is to stablish a theory of Baer invariants, associated to certain types of varieties of (pre)crossed modules, and to obtain a five term exact sequence and 'the basic theorem' of Stallings in this setting. Our method, which extends results of [4] and aspects of [9], gives a systematic treatment for several cases.

The category of precrossed modules, $\mathcal{Gr}\#$, is a Burgin category [2]. Also it is a (co)complete category, has free objects with respect to the forgetful functor, U, to the category of maps of $\mathcal{Set}$ and they are $\mathcal{E}$ – projective ($\mathcal{E}$ being the class of U – split surjections of $\mathcal{Gr}\#$). Proofs of these facts are routine. Consider the subcategory, $\mathcal{Gr}\#\text{-}G$, of precrossed G – modules over a group G. The arguments used here for $\mathcal{Gr}\#$ can be easily adequated to the subcategory $\mathcal{Gr}\#\text{-}G$ (which has not a zero object!), $\mathcal{E}$ being now the class of all surjections. As a consequence, following [5] and [6], if $\mathcal{V}$ is a variety of $\mathcal{Gr}\#\text{-}G$ or $\mathcal{Gr}\#$, then the ($\mathcal{E}$ –)Baer invariants, associated to $\mathcal{V}$, can be defined in the way

$$\Delta^{\mathcal{E}}V(X) = (R \cap V(F))/V_1(R,F), \qquad D^{\mathcal{E}}V(C) = V(F)/V_1(R,F),$$

$R \rightarrowtail F \twoheadrightarrow X$ being an $\mathcal{E}$ – projective presentation in $\mathcal{Gr}\#\text{-}G$ or $\mathcal{Gr}\#$.

THEOREM 1. *If $\mathcal{V}$ is any variety of $\mathcal{Gr}\#\text{-}G$ or $\mathcal{Gr}\#$ and* $X' \rightarrowtail X \twoheadrightarrow X''$ is an *$\mathcal{E}$ – exact sequence, then we have a commutative diagram with exact rows:*

$$\begin{array}{ccccc} \mathrm{Ker}\ D^{g}V(\alpha) & \rightarrowtail & D^{g}V(X) & \xrightarrow{\alpha} & D^{g}V\,(X'') \\ \gamma\downarrow & & \downarrow & & \downarrow \\ X' & \rightarrowtail & X & \twoheadrightarrow & X'' \end{array}$$

Therefore, there exists an exact and natural sequence

$$\Delta^{g}V(X) \longrightarrow \Delta^{g}V(X'') \longrightarrow X'/\mathrm{Im}\,\gamma \longrightarrow X/V(X) \twoheadrightarrow X''/V(X'').$$

Proof. It is a simple translation of the proof of [6, theorem 9], having into account the properties of $\mathcal{G}r\#$, mentioned above. In the case of $\mathcal{G}r\#\text{-}G$, an obvious adequacy of these proof gives the result. ∎

The five term exact sequence, in the usual form, can be obtained for those varieties of $\mathcal{G}r\#$ for which $\mathrm{Im}\,\gamma = V_1(C' \to G', C \to G)$. It is enough that V_1 preserves surjections (this fact has been proved in [5, theorem 2.1] for varieties of $\Omega-$groups). In our case it is a consequence of explicit computations of the functor V_1, which we obtain for several types of varieties in $\mathcal{G}r\#$ (theorem 2, below). These are, first, the variety, $\mathcal{C}\mathcal{G}r\#$, of crossed modules. Also, for a variety, $\mathcal{V}$, of ordinary groups, consider the following variety of $\mathcal{G}r\#$

$$\mathcal{V}_{\to} := \{(C \to G) \in \mathcal{V}\# : \text{the action of } G \text{ on } C \text{ is trivial}\}$$

So $\mathcal{A}b_{q\to}$ is the variety of abelian crossed modules of exponent q $(q=0,1,\ldots)$.

Let $(C \to G) \in \mathcal{G}r\#$, $C' < C$ and $G' < G$. Let us denote $\overline{C'G'}$ the subgroup of C generated by: $c'^{g}c'^{-1}$, $c' \in C'$, $g' \in G'$, and let $<,>$ denote the Peiffer commutator.

THEOREM 2. *Let $C' \longrightarrow G'$ be a normal subobject of $C \xrightarrow{\partial} G \in \mathcal{G}r\#$. Then*
a) For the variety $\mathcal{C}\mathcal{G}r\#$ of $\mathcal{G}r\#$ and assuming $G' = 1$, we have

$$V_1(C' \to 1, C \to G) = (<C',C> \cdot <C,C'> \to 1).$$

b) For the variety $\mathcal{V}_{\to}$ and assuming that $(C \longrightarrow G) \in \mathcal{C}\mathcal{G}r\#$, we have

$$V_1(C' \to G', C \to G) = (\overline{C'G} \cdot \overline{CG'} \cdot V(C',C) \to [\partial C', G] \cdot [\partial C, G'] \cdot V(G'.G)).$$

Proof. The functor V_1, as defined in [6] for $\Omega-$groups, is the least satisfying the properties (i) and (ii) of [5, theorem 2.1]. Therefore, once we prove that the right members are normal subobjects, it is enough to prove that the "property (ii) of [5, theorem 2.1]" = (p) is verified:

a) $C_1 = <C',C> \cdot <C,C'>$ is generated by $<x,y>^{-1}<xx',yy'>$, $x,y \in C$, $x',y' \in C'$. In fact:

$$<x,y>^{-1}<xx',yy'> = <x,y>^{-1}x<x',xx'>x^{-1}<x,yy'> =$$
$$y^{\partial x}xy^{-1}<x',yy'>x^{-1}<x,y>y^{\partial x}<x,y'>(y^{\partial x})^{-1} =$$
$$y^{\partial x}(xy^{-1}<x',yy'>yx^{-1}) \cdot <x,y'>(y^{\partial x})^{-1} \in <C',C> \cdot <C,C'>.$$

Therefore C_1 verifies (p).

If $C''' \to 1$ is a normal subobject verifying this property, let $F \to G$ be the free precrossed $G-$module [10] on $\{c,c'\} \to G$ and $\phi,\psi: F \to G$ such that $\phi c = cc'$, $\phi c' = c'$; $\psi c = c$,

$\psi c' = 1$. Then $<c,c'> = <cc',c'>\cdot<c,1>^{-1} = \phi<c,c'>\cdot\psi<c,c'>^{-1} \in C''$.

b) By [5, theorem 2.1] $V_1(C',C) = V(C',C)$ and $V_1(G',G) = V(G',G)$. If (ϕ,α), (ψ,β): $(D \rightarrow H) \longrightarrow (C \rightarrow G)$ are morphisms in $\mathscr{C}\mathscr{G}r\#$, coequal to $(C/C' \rightarrow G/G')$, then

$$(\psi d^{-1})^{\alpha g}\cdot\phi(d^g d^{-1})\cdot\psi(d^g d^{-1})^{-1}\cdot(\psi d)^{\alpha g} = [(\psi d^{-1}\phi d)^{\alpha g}\cdot\phi d^{-1}\psi d]\cdot[(\psi d^{-1})^{\beta g}(\psi d)^{\alpha g}] \in \overline{C'G}\cdot\overline{CG'}.$$

Thus (p) is verified.

If $C'' \rightarrow G''$ is a normal subobject of $C \rightarrow G$ verifying (p) and $c'^g c'^{-1} \in \overline{C'G}$, by considering the free crossed module on the map $\{c'\} \rightarrow \{g, \partial c'\}$, we obtain that $c'^g c'^{-1} \in C''$. Analogously, $\overline{CG'} \subseteq C''$. Let $\phi,\psi: D \longrightarrow C$ be homomorphisms of groups, coequal to C/C'. Since $(C \rightarrow G) \in \mathscr{C}\mathscr{G}r\#$, the morphisms $(\phi,\partial\phi),(\psi,\partial\psi): (D \rightarrow D) \longrightarrow (C \rightarrow G)$ are coequal to $(C/C' \rightarrow G/G')$. Then $V(C',C) \subseteq C''$. Analogously $V(G',G) \subseteq G''$. ∎

COROLLARY 3. *Let V_1 be the functor, computed in theorem 2, corresponding to each of the varieties considered in the following.*

a) The variety $\mathscr{C}\mathscr{G}r\#$-$G$ in $\mathscr{G}r\#$-G. If $C' \rightarrowtail C \twoheadrightarrow C''$ is an exact sequence in $\mathscr{G}r\#$-G, then there exists a five term exact sequence

$$\Delta V(C) \longrightarrow \Delta V(C'') \longrightarrow C'/V_1(C',C) \longrightarrow C/<C,C> \twoheadrightarrow C''/<C'',C''>.$$

b) The variety $\mathscr{V}$ in $\mathscr{C}\mathscr{G}r\#$. If $1 \longrightarrow (C' \rightarrow G') \longrightarrow (C \rightarrow G) \longrightarrow (C'' \rightarrow G'') \longrightarrow 1$ is an $\mathscr{E}$-exact sequence in $\mathscr{C}\mathscr{G}r\#$, then there exists a five term exact sequence

$$\Delta^{\mathscr{E}}V(C \rightarrow G) \longrightarrow \Delta^{\mathscr{E}}V(C'' \rightarrow G'') \longrightarrow (C' \rightarrow G')/V_1(C' \rightarrow G', C \rightarrow G) \longrightarrow$$
$$\longrightarrow (C \rightarrow G)/V(C \rightarrow G) \twoheadrightarrow (C'' \rightarrow G'')/V(C'' \rightarrow G'') \twoheadrightarrow 1.$$

Proof. Since V_1, as computed in theorem 2, preserves surjections, it follows that $\operatorname{Im}\gamma = V_1(X',X)$ in any case, analogously to [6]. ∎

COROLLARY 4. (Basic theorem of Stallings for crossed modules) *Let $f: M \longrightarrow N$ be a morphism in any of the categories $\mathscr{G}r\#$-G or $\mathscr{C}\mathscr{G}r\#$ and consider any of the varieties in corollary 3. Assume that, in the case of $\mathscr{C}\mathscr{G}r\#$, every term of the lower V-central series $(V^{i+1}(M) = V_1(V^i(M),M))$ of M and N is $\mathscr{E}$-allowable. If f induces an isomorphism, $M/V(M) \longrightarrow N/V(N)$, and a surjection, $\Delta^{\mathscr{E}}V(M) \longrightarrow \Delta^{\mathscr{E}}V(N)$, then it induces isomorphisms, $M/V^i(M) \longrightarrow N/V^i(N)$, $i \geqslant 1$.*

Proof. Having into account corollary 3 it is an easy translation of the proof given in [11]. ∎

The proposition of p. 332 and theorem 1 of [4] are particular cases of our corollaries 3 and 4, respectively. The lower V-central series has been considered in [9] for the variety of abelian crossed modules. These results also extend to the corresponding ones for ordinary groups [11] (when a group G is considered as a crossed module $G \rightarrow 1$), since $F \rightarrow 1$ is an $\mathscr{E}$-projective crossed module if F is free.

We also have calculated, for the variety $\mathscr{V}$, the V-centre of a crossed module: $V^*(C \rightarrow G) = (C^G \cap_q \zeta(C) \rightarrow (\bigcap_C G_c) \cap V^*G)$, where $\mathscr{V} \cap \mathscr{A}\mathscr{b} = \mathscr{A}\mathscr{b}_q$, V^*G is the V-centre (marginal subgroup) of G, ${}_q\zeta(C)$ is the $q-$centre, C^G is the fixed group of C and G_c is the isotropy group. When we take the variety, $\mathscr{A}\!\mathscr{b}_{\to}$, of abelian crossed modules, this notion coincides with the centre constructed in [9] and with the one of [7]. The same holds also for the commutator, V_1, corresponding to $\mathscr{A}\!\mathscr{b}_{\to}$. Thus, $V-$nilpotency and $V-$solvability can be extended to crossed modules.

Similar results are obtained for the category, $\mathscr{L}ie\#$, of (pre)crossed modules of Lie algebras, by a routine translation.

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