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On Some Fréchet Interpolation Spaces

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A basic knowledge about locally convex spaces can be obtained from [9], about operator ideals from [11], and about both together from [8]. We follow those books concerning notation.

Let E be a locally convex space and $\mathscr{A}$ an operator ideal. We say that E is generated by $\mathscr{A}$ if for any neighborhood of 0, U there is another neighborhood of 0, $V \subset U$, such that the canonical linking map $\hat{T}_{VU} \in \mathscr{A}(\hat{E}_V, \hat{E}_U)$.

The class of all locally convex spaces generated by $\mathscr{A}$ is denoted by Groth($\mathscr{A}$). It is a standing problem what can be said about the structure of E recovered from its pertenece to a class Groth($\mathscr{A}$). Several contributions to this problem are in previous papers of the author [3], [4], [5].

In this paper a special case of the problem is considered: starting with a continuous injection $I: X \longrightarrow Y$ between Banach spaces, we form a reduced projective limit of real interpolation spaces, obtaining in this way a Fréchet (non Banach) space, which we call $[X,Y]_{(\theta),q}$. We study relationships between $I \in \mathscr{A}$ and $[X,Y]_{(\theta),q} \in$ Groth($\mathscr{A}$) for several choices (unconditionally converging, weakly compact, compact, completely continuous and nuclear operators) of the ideal $\mathscr{A}$.

Let us consider a couple of Banach spaces X, Y linked by a continuous injection with dense range $I: X \longrightarrow Y$. The pair (X,Y) can be considered an interpolation couple. See [2] for basic facts about interpolation. When $0 < \theta < 1$, and $1 \leqslant q \leqslant +\infty$, the real interpolation method gives a Banach intermediate space $[X,Y]_{\theta,q}$. When $\theta' < \theta$, the inclusion I induces a continuous injection $I_{\theta'\theta}: [X,Y]_{\theta',q} \longrightarrow [X,Y]_{\theta,q}$.

Let us fixe q, and take the projective limit

$$[X,Y]_{(\theta),q} := \varprojlim I_{\theta'\theta}\Big[[X,Y]_{\theta,q}\Big]$$

with respect to the maps $I_{\theta'\theta}$.

The space $[X,Y]_{(\theta),q}$ is, to an extent, a generalization of some interesting spaces. If, for instance, we take $X = \ell_p$ with the natural inclusion $\ell_p \longrightarrow \ell_\infty$, then $[\ell_p, \ell_\infty]_{(\theta),q} = \cap_{\epsilon>0} \ell_{p+\epsilon}$.

The question to be answered in this paper is to what extent "$[X,Y]_{(\theta),q} \in$ Groth($\mathscr{A}$)" and "$I \in \mathscr{A}$" are equivalent. Firstly notice that if $[X,Y]_{(\theta),q} \in$ Groth($\mathscr{A}$), then not only $I \in \mathscr{A}$ but even $I \in \mathscr{A}^n$ for all n due to an obvious factorization

$$\begin{array}{ccc} X & \xrightarrow{I} & Y \\ & \searrow \nearrow & \\ & [X,Y]_{(\theta),q} & \\ \swarrow & & \searrow \\ [X,Y]_{\theta',q} & \longrightarrow & [X,Y]_{\theta,q} \end{array}$$

wich implies that

LEMMA 1. $[X,Y]_{(\theta),q} \in \mathscr{A} \Rightarrow I \in \cap_{n\in \mathbb{N}} \mathscr{A}^n$.

For this reason we should choose idempotent operators ideals $\mathscr{A}$.

When $\mathscr{A} = \mathscr{W}$ (weakly compact operators) or $\mathscr{K}$ (compact operators) then it is a simple rewarding of some results of Beauzamy, joint to the "reiteration theorem", that

PROPOSITION 2. *Let $0 < \theta < 1$, $1 < q < +\infty$. $[X,Y]_{(\theta),q}$ is a Schwartz (resp. an infra–Schwartz) space if and only if $I \in \mathscr{K}$ (resp. $I \in \mathscr{W}$).*

If we choose $\mathscr{A} = \mathscr{U}$ (unconditionally converging operators, that is, those sending weakly summable sequences into summable ones), then we find some troubles. Recall that a continuous operator $T : X \longrightarrow Y$ acting between Banach spaces is in $\mathscr{U}$ if and only if it is not an isomorphism when restricted to any subspace of X isomorphic to c_0; this implies, in particular that a Banach space X does not contain a copy of c_0 if and only if $\mathrm{id}(X) \in \mathscr{U}$. This later result is still true for Fréchet spaces and we give in the paper one of the several possible proofs for this fact (see also [7]).

From this we derive

PROPOSITION 3. *If Y does not contain c_0 then $[X,Y]_{(\theta),q} \in$ Groth($\mathscr{U}$).*

In particular, this implies that:

COROLLARY 4. *If Y does not contain c_0, then $[X,Y]_{(\theta),q}$ does not contain c_0.*

In the Banach space setting an analogous result was proved by Levy [10]. Since a Fréchet space not containing c_0 can have associated Banach spaces containing c_0 (just consider Köthe's example of a Fréchet Montel not Schwartz echelon space of order 0 (see [8])), Levy's result does not subsumes Corollary 4. It remains however to be know whether the hypothesis " Y does not contain c_0" is necessary.

Finally, if we consider the operator ideal $\mathscr{A} = \mathscr{B}$ (completely continuous operators, that is, those sending weakly null sequences into norm null sequences), a trivial example, the canonical inclusion $\ell_1 \longrightarrow \ell_\infty$, shows that $I \in \mathscr{B}$ does not imply $[X,Y]_{(\theta),q} \in \mathscr{B}$.

For the sake of completeness we also consider in the paper the case of $\mathscr{A} = \mathscr{N}$ (nuclear operators, that is, those admitting a representation of the form $T = \Sigma_n \lambda_n x_n^* \otimes y_n$, with (x_n^*) and (y_n) bounded sequences and $(\lambda_n) \in \ell_1$). This operator ideal is not idempotent. The ideal $\mathscr{N}_0 = \cap_{n\in \mathbb{N}} \mathscr{N}^n$ receives the name of ideal of strongly nuclear operators, and operators in $\mathscr{N}_0$ are characterized by admitting a representation as above but with $(\lambda_n) \in \cap_{p>0} \ell_p$. We find that:

PROPOSITION 5. $[X,Y]_{(\theta),q}$ *is a nuclear space if and only if* $I \in \mathcal{N}_0$.

This we prove making a simple calculation with entropy numbers:

PROPOSITION 6. *If* I *belongs to some entropy ideal* $\mathscr{E}_p$, *then* $I_{\theta'\theta} \in \mathscr{E}_p/\theta-\theta'$.

It would be interesting to carefully check if a similar result like this holds for real interpolation with functional parameter. The behaviour of entropy numbers under real interpolation with functional parameter was obtained in [6].

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