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Pontryagin Duality and Strong Duality

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One of the important theorems that make sense in the commutative harmonic analysis without the asumption of the local compactness of the group is the Pontryagin – van Kampen duality theorem. This theorem is known to be true e. g. for Banach spaces [10], products of locally compact groups [7] or additive subgroups and quotients of nuclear Fréchet spaces [2].

For a locally compact group G, the duality between G and its dual group $G^{\wedge}$ is inherited by closed subgroups and Hausdorff quotients. This duality is called strong duality and it is a stronger property than duality. Some new facts about the duality of Pontryagin and strong duality are given in this paper.

The topological groups under consideration will be Hausdorff, abelian and additive, denoting the neutral element by 0. The additive groups of integers and of reals will be denoted by $\mathbb{Z}$ and $\mathbb{R}$ respectively.

By a character of a group G, we mean a continuous homomorphism of G into the group $T:=\mathbb{R}/\mathbb{Z}$. We shall identify T with the interval $(-1/2,1/2]$. The set of all characters of G, with addition defined pointwise, is again an abelian group called the dual group of G, or the character group of G. It will be denoted by $G^{\wedge}$. Various topologies can be considered on $G^{\wedge}$; we will consider $G^{\wedge}$ endowed with the topology of compact convergence, usually called the compact – open topology.

The polar set of a subset A of G is defined by $A^{\circ} = \{\chi \in G^{\wedge} : 1/4 \geqslant \chi(A)\}$. It is known [9] that polars of compact subsets of G form a base at zero in $G^{\wedge}$. Another interesting fact about polars is that polars of neighbourhoods of zero in G are compact subsets of $G^{\wedge}$.

If A is a subgroup of G, then $A^{\circ} = \{\chi \in G^{\wedge} : \chi_{|A} \equiv 0\}$. Thus A° is a closed subgroup of $G^{\wedge}$; we call it the annihilator of A. If B is a subgroup of $G^{\wedge}$, then we denote B° the set $\{g \in G : \chi(g)=0 \text{ for all } \chi \in B\}$.

We say that the group G has sufficiently many characters if for each $g \neq h$ in G, there is some $\chi \in G^{\wedge}$ with $\chi(g) \neq \chi(h)$. If H is a subgroup of a topological group G, we say that H is dually closed in G if for each $g \notin H$, there is some $\chi \in H^{\circ}$ with $\chi(g) \neq 0$. This is equivalent to the assertion that the Hausdorff quotient G/H has sufficiently many characters (note that dually closed subgroups are closed). Next, we say that H is dually embedded in G if each character of H can be extended to a character of G.

Let us recall shortly basic facts concerning the Pontryagin – van Kampen duality theorem for locally compact abelian groups (LCA groups). The proofs can be found e. g. in [8, pp. 84,90,91].

THEOREM 1. *Let G be a LCA group. Then $G^\wedge$ is also a LCA group and*
a) The evaluation map $\alpha_G : G \longrightarrow (G^\wedge)^\wedge$ $((\alpha(g)(\chi) = \chi(g)$, for $g \in G$, $\chi \in G^\wedge)$ is a topological isomorphism of G onto $G^{\wedge\wedge} := (G^\wedge)^\wedge$.
b) If H, F are closed subgroups of G and $G^\wedge$ respectively, then they are dually closed and dually embedded.
c) The canonical mappings $G^\wedge / H^\circ \longrightarrow H^\wedge$, $(G/H)^\wedge \longrightarrow H^\circ$, $G/F^\circ \longrightarrow F^\wedge$ and $(G^\wedge/F)^\wedge \longrightarrow F^\circ$ are topological homomorphisms.

We say that G is a reflexive group if it satisfies *a)*. If G satisfies all the assertions of the above proposition it is called strongly reflexive using the terminology of [5].

We give now a characterization of strongly reflexive groups.

PROPOSITION 1. *A reflexive group is strongly reflexive if and only if every closed subgroup and every Hausdorff quotient group of G and of $G^\wedge$ is reflexive.*

The proof of the necessity can be found in [5] and the proof of the sufficiency will be given in [6].

It is a known fact that there exist strongly reflexive groups which are not locally compact; for instance, products of a compact group with a countable product of copies of $\mathbb{R}$ and $\mathbb{Z}$ [5], or nuclear Fréchet spaces [2]. But strongly reflexivity is a strictly stronger property than reflexivity, that is to say, reflexivity is not in general inherited by closed subgroups or Hausdorff quotients. In fact, every infinite dimensional Banach space considered in its group structure is a reflexive group [10] and contains a discrete subgroup K which is not dually closed and consequently G/K is not reflexive [1].

On the other hand, arbitrary products of reflexive groups are reflexive [7] and the product $\omega\mathbb{R} \times \mathbb{R}^\omega$ is not strongly reflexive [3]. (Here $\omega\mathbb{R}$ and $\mathbb{R}^\omega$ represent the countable direct sum and the countable product of real lines respectively).

Let A be an open subgroup of G; then A is dually closed and dually embedded in G [9]. Moreover we have the following property

PROPOSITION 2. *If A is an open subgroup of G, the canonical mappings $G^\wedge / A^\circ \longrightarrow A^\wedge$ and $(G/A)^\wedge \longrightarrow A^\circ$, are topological isomorphisms.*

The proof will be given in [4].

Venkataraman [11] proved that if G is reflexive then open subgroups of G are reflexive too. Moreover we have that converse is also true.

THEOREM 2. *Let A be an open subgroup of G. Then the group G is reflexive (resp. strongly reflexive) if and only if A is reflexive (resp. strongly reflexive).*

The proof will be given in [4].

Finally owing to the fact that a closed subgroup A of a reflexive group G is open if and only if its annihilator A° is a compact subgroup of $G^\wedge$, a dual of theorem 2 can be obtained for compact subgroups.

COROLLARY 1. *a) If K is a compact subgroup of G and the quotient group G/K is reflexive, then the group G is reflexive. Moreover if K is dually closed and G is reflexive, G/K is also reflexive.*
b) If K is a compact subgroup of a reflexive group G, then G is strongly reflexive if and only if G/K is strongly reflexive.

Remark. Theorem 2 could be applied to prove that the product of a strongly reflexive group and a discrete group is strongly reflexive.

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