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Weak Moduli of Convexity [1]

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1. INTRODUCTION

Let E be a real normed linear space with unit ball B and unit sphere S. The classical modulus of convexity of J.A. Clarkson [2]

$$\delta_E(\epsilon)=\inf\{1-\tfrac{1}{2}\|x+y\|:x,y\in B,\ \|x-y\|\geqslant\epsilon\}\qquad(0\leqslant\epsilon\leqslant2)$$

is well known and it is at the origin of a great number of moduli defined by several authors. Among them, D.F. Cudia [3] defined the directional, weak and directional weak modulus of convexity of E, respectively, as

$$\overline{\delta}_E(\epsilon,g)=\inf\{1-\tfrac{1}{2}\|x+y\|:x,y\in B,\ g(x-y)\geqslant\epsilon\}$$

$$\delta_E(\epsilon,f)=\inf\{1-\tfrac{1}{2}f(x,y):x,y\in B,\ \|x-y\|\geqslant\epsilon\}$$

$$\delta_E(\epsilon,f,g)=\inf\{1-\tfrac{1}{2}f(x,y):x,y\in B,\ g(x-y)\geqslant\epsilon\}$$

where $0\leqslant\epsilon\leqslant2$ and $f,g\in S'$ (unit sphere of the topological dual space E').

D.F. Cudia [3] has shown the close connection existing between these moduli and various differentiability conditions of the norm in E'.

In this note we study these moduli from a different point of view, then we analyze some of its properties and we see that it is possible to characterize inner product spaces by means of them.

2. RESULTS

From the Hahn–Banach theorem we get

PROPOSITION 1. *For every* $f,g\in S'$ *and every* $0\leqslant\epsilon\leqslant2$,

i) $\delta_E(\epsilon,f)=\inf_{g\in S'}\delta_E(\epsilon,f,g)$.

ii) $\overline{\delta}_E(\epsilon,g)=\inf_{f\in S'}\delta_E(\epsilon,f,g)$.

iii) $\delta_E(\epsilon)=\inf_{f\in S'}\delta_E(\epsilon,f)=\inf_{g\in S'}\overline{\delta}_E(\epsilon,g)$.

Our second result simplifies the definitions of the moduli in question.

PROPOSITION 2. *For every* $f,g\in S'$ *and every* $0\leqslant\epsilon\leqslant2$,

[1] This note is part of A. Ullán's doctoral dissertation [4] and was explained in the "II Congreso de Análisis Funcional" held at Jarandilla de la Vera (Spain), 1990.

i) $\delta_E(\epsilon,f) = \inf\{1-\frac{1}{2}f(x,y) : x,y \in S,\ \|x-y\| = \epsilon\}$.

ii) $\bar{\delta}_E(\epsilon,g) = \inf\{1-\frac{1}{2}\|x+y\| : x,y \in S,\ g(x-y) = \epsilon\}$.

iii) $\delta_E(\epsilon,f,g) = \inf\{1-\frac{1}{2}f(x,y) : x,y \in S,\ g(x-y) = \epsilon\}$.

Concerning continuity and convexity properties we have,

PROPOSITION 3. *For every* $f,g \in S'$,

i) The function $\epsilon \longrightarrow \delta_E(\epsilon,f,g)$ *is convex in* $[0,2]$.

ii) The functions $\epsilon \longrightarrow \delta_E(\epsilon,f)$ *and* $\epsilon \longrightarrow \bar{\delta}_E(\epsilon,g)$ *are continuous in* $[0,2)$.

Finally, with regard to inner product spaces, we have obtained the following results.

PROPOSITION 4. *Let E be a real inner product space. Then for* $f,g \in S'$ *and every* $0 \leqslant \epsilon \leqslant 2$,

$$\delta_E(\epsilon,f) = \bar{\delta}_E(\epsilon,g) = 1-(1-\epsilon^2/4)^{\frac{1}{2}}.$$

Now, having in mind Proposition 1, it is obvious that if E is an inner product space, then the inequality

$$\delta_E(\epsilon,f,g) \geqslant 1-(1-\epsilon^2/4)^{\frac{1}{2}}$$

holds for every $f,g \in S'$ and every $0 \leqslant \epsilon \leqslant 2$.

However, in general, the inequality expressed above cannot be replaced by an equality. In fact, it is easy to see that in every normed linear space E

$$\delta_E(\epsilon,f,f) \geqslant \epsilon/2 > 1-(1-\epsilon^2/4)^{\frac{1}{2}}$$

for every $f \in S'$ and every $0 < \epsilon < 2$.

On the other hand, we have

PROPOSITION 5. *Let E be an inner product space, and let* $f,g \in S'$ *and* $0 < \epsilon < 2$. *Then*

$$\delta_E(\epsilon,f,g) = 1-(1-\epsilon^2/4)^{\frac{1}{2}}$$

if and only if f is orthogonal to g.

The above formulae suggested us the possibility to obtain with these moduli characterizations of inner product spaces similar to the ones obtained by J. Alonso and C. Benítez [1] with the modulus of Clarkson.

Let D be the countable and dense subset of the interval (0,2) defined in [1] as

$$D = \{2\cos\tfrac{k\pi}{2n} : n = 1,2,3\ldots,\ k = 1,2,\ldots,n-1\},$$

PROPOSITION 6. *Let* $\epsilon \in (0,2)\backslash D$. *Then any one of the following properties is a sufficient condition to be E an inner product space.*

i) $\delta_E(\epsilon,f) \geqslant 1-(1-\epsilon^2/4)^{\frac{1}{2}}$, *for every* $f \in S'$.

ii) $\bar{\delta}_E(\epsilon,g) \geqslant 1-(1-\epsilon^2/4)^{\frac{1}{2}}$, *for every* $g \in S'$.

In the same way as J. Alonso and C. Benítez [1] did for the modulus $\delta_E(\epsilon)$, we conjecture that the last Proposition is true without restriction on $\epsilon \in (0,2)$ if $\dim E$ is greater than two.

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