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A Real Inversion Formula for the Kratzel's Generalized Laplace Transform

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E. Kratzel [5] introduced a generalized Laplace transformation defined by

$$\mathscr{L}_\nu^{(\rho)}(f)(y) = \int_0^\infty \lambda_\nu^{(\rho)}(xy) f(x)\,\mathrm{d}x, \quad x>0 \tag{1}$$

where

$$\lambda_\nu^{(\rho)}(z) = \frac{(2\pi)^{(\rho-1)/2}\sqrt{\rho}}{\Gamma[\nu+1-(1/\rho)]} \cdot \left[\frac{z}{\rho}\right]^{\rho\nu} \cdot \int_1^\infty (t^\rho-1)^{\nu-(1/\rho)} e^{-zt}\,\mathrm{d}t, \quad z>0$$

for $\rho \in \mathbb{N}$ and $\nu > -1+1/\rho$. He studied in a serie of papers ([5], [6] and [7]) the main classical properties of $\mathscr{L}_\nu^{(\rho)}$. Recently, we have completed the investigations of E. Kratzel. In [3] we shown new real inversion formulas for $\mathscr{L}_\nu^{(\rho)}$ and in [4] we established integrability properties for the transformation (1). The $\mathscr{L}_\nu^{(\rho)}$ transformation was defined in spaces of generalized functions in [1] and [2].

Our purpose in this note is to prove a real inversion formula for $\mathscr{L}_\nu^{(\rho)}$ when $\rho \in \mathbb{R}$ and $\rho \geqslant 1$.

Throughout this paper for every $\alpha>0$ we shall denote by D_k^α the Weyl fractional derivative defined by (see [8])

$$(D_k^\alpha f)(x) = D^n K^{n-\alpha}(f)(x)$$

where $K^{n-\alpha}(f)(x) = \frac{1}{\Gamma(n-\alpha)}\int_x^\infty (t-x)^{n-\alpha-1} f(t)\,\mathrm{d}t$, for $x>0$, and $n\in\mathbb{N}$ such that $n-1 \leqslant \alpha < n$.

We need firstly establish the following result.

LEMMA 1. *If* $\rho \geqslant 1$ *and* $\nu > -1+1/\rho$ *then*

$$z^{\rho\nu-1} D_k^{\rho-1} z^{1-\rho\nu} D \lambda_\nu^{(\rho)}(z) = (-1)^{n+1} \lambda_\nu^{(\rho)}(z) \tag{2}$$

where $n\in\mathbb{N}$ such that $n \leqslant \rho < n+1$.

Proof. To show (2) it is sufficient to use that $K^\alpha(e^{-zt})(x) = z^{-\alpha} e^{-zt}$, for every $\alpha>0$ and $z>0$. ∎

The inversion formula presented in the next theorem reduces to the one proved in [3] when $\rho\in\mathbb{N}$. Moreover our inversion formula generalizes the one due to D.V. Widder [9] for the Laplace transformation.

THEOREM 1. *Let $\rho \geqslant 1$ and $n \in \mathbb{N}$ such that $n \leqslant \rho < n+1$. Assume that $(n+1)/\rho - 1 < \nu < 1/\rho$ and that $f(x)$, $0 < x < \infty$, is a complex function satisfying*

i) $f(x) \in L_1(R^{-1}, R)$, *for every* $R > 1$,

ii) $f(x)\, e^{-cx} \in L_1(1,\infty)$, *for some* $c > 0$,

iii) $f(x)\, x^r \in L_1(0,1)$, *for some* $r \in (\rho^{-1} + (\rho-1)\nu - 2, \rho^{-1} + (\rho-1)\nu - 1)$, *and*

iv) $\int_t^s |\Phi_y(u) - \Phi_y(t)|\, du = o(|s-t|)$, *as* $s \longrightarrow t^+$, *where*

$$\Phi_y(u) = |\, u^{\rho^{-1}+(\rho-1)\nu-1} f(u) - y^{\rho^{-1}+(\rho-1)\nu-1} f(y)\,|, \text{ for every } y > 0,$$

then

$$\lim_{k\to\infty} \frac{\rho^{\rho\nu+(1/2)}\Gamma[k+\frac{\nu+1}{\rho}-\rho^{-2}+1]\,(\rho k)^{2-(\rho-1)\nu+\rho k-(1/\rho)}}{(2\pi)^{(\rho-1)/2}\Gamma[\rho k+2+\nu-\rho^{-1}]\,\Gamma[k+\frac{\rho-1}{\rho}\nu-\rho^{-2}+(2/\rho)]} \cdot$$

$$\cdot\, (-1)^{(n+1)k}\, y^{-\rho k-1}\, B^k_{\nu,\rho}[F(z)]_{/z=\rho k/y} = f(y),$$

for every $y > 0$, *where* $B_{\nu,\rho} = z^{\rho\nu-1} D_k^{\rho-1} z^{1-\rho\nu} D$ *and* $F(y) = \mathscr{L}_\nu^{(\rho)}(f)(y)$.

Sketch of the proof. According to well-known abelian theorems for the Laplace transform (see [9, p.180]) we get

$$B^k_{\nu,\rho}F(z) = \int_0^\infty B^k_{\nu,\rho,z}(\lambda_\nu^{(\rho)}(zx))\, f(x)\, dx,$$

when $z > z_0$, for some $z_0 > 0$, and $k \in \mathbb{N}$. Hence, by vitue of (2), it follows that

$$B^k_{\nu,\rho}F(z) = (-1)^{(n+1)k} \int_0^\infty x^{\rho k} \lambda_\nu^{(\rho)}(zx)\, f(x)\, dx,$$

for $z > z_0$ and $k \in \mathbb{N}$. Moreover

$$\int_0^\infty x^{\rho k-(\rho-1)\nu+1-(1/\rho)} \lambda_\nu^{(\rho)}(x\rho k/y)\, dx = (y/\rho k)^{\rho k+2-(\rho-1)\nu-(1/\rho)} \cdot M_{\nu,\rho,k}$$

where

$$M_{\nu,\rho,k} = \frac{(2\pi)^{(\rho-1)/2}\Gamma[\rho k+2+\nu-\rho^{-1}]\,\Gamma[k+\frac{\rho-1}{\rho}\nu-\rho^{-2}+(2/\rho)]}{\rho^{\rho\nu+(1/2)}\Gamma[k+\frac{\nu+1}{\rho}-\rho^{-2}+1]}$$

for every $y > 0$. Then, for every $y > 0$ and $k > k(y)$, for some $k(y) \in \mathbb{N}$,

$$H_{\nu,\rho,k}(y) = (\rho k)^{\rho k+2-(\rho-1)\nu-(1/\rho)} M^{-1}_{\nu,\rho,k} (-1)^{(n+1)k} y^{-\rho k-1} B^k_{\nu,\rho}[F(z)]_{/z=\rho k/y} - f(y) =$$

$$= (\rho k)^{\rho k+2-(\rho-1)\nu-(1/\rho)} M^{-1}_{\nu,\rho,k} y^{-\rho k-1} \cdot$$

$$\cdot \int_0^\infty x^{\rho k-(\rho-1)\nu+1-(1/\rho)} \lambda_\nu^{(\rho)}(x\rho k/y)\,[x^{(1/\rho)+(\rho-1)\nu-1} f(x) - y^{(1/\rho)+(\rho-1)\nu-1} f(y)]\, dx.$$

By invoking the asymptotic behaviour of $\lambda_\nu^{(\rho)}(z)$ near the origin and the infinity we can obtain

$$|H_{\nu,\rho,k}(y)| \leqslant M y^{1-(\rho-1)\nu-(1/\rho)} \frac{\Gamma[k+\frac{\nu+1}{\rho}-\rho^{-2}+1]\,\Gamma[\rho k+1]}{\Gamma[\rho k+2+\nu-\rho^{-1}]\,\Gamma[k+\frac{\rho-1}{\rho}\nu-\rho^{-2}+(2/\rho)]} \cdot$$

$$\cdot \frac{1}{\Gamma(\rho k+1)}\left[\frac{\rho k}{y}\right]^{\rho k+1}\int_0^\infty x^{\rho k}\, e^{-\rho k x/y}\, |x^{(1/\rho)+(\rho-1)\nu-1}f(x)-y^{(1/\rho)+(\rho-1)\nu-1}f(y)|\, dx$$

for a certain $M>0$.

Finally by using [9, Theorem 3.a, p.283] we establish the theorem. ∎

In [2] we define the $\mathscr{L}_\nu^{(\rho)}$ transform of distributions of compact support when $\rho \in \mathbb{N}$ by employing the kernel method. In a similar way we can investigate the $\mathscr{L}_\nu^{(\rho)}$ transformation on $E'(I)$ for $\rho \geqslant 1$ obtaining a distributional version of Theorem 1.

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