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A Generalized Hankel Transformation on the Spaces $F_{p,\mu}$

MANUEL T. FLORES AND JORGE J. BETANCOR

Dpto. de Análisis Matemático, Univ. La Laguna, Tenerife, Spain

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E. Kratzel [2] introduced the function

$$J_\nu^{(n)}(x) = \frac{2}{\sqrt{\pi}\,\Gamma(\nu+1-1/n)}(x/2)^{n\nu/2}\int_0^1 (1-t^n)^{\nu-1/n}\cos(xt)\,dt,$$

for $x>0$, $\nu>-1+1/n$ and $n\in\mathbb{N}^*=\mathbb{N}-\{0\}$.

A generalization of the Hankel transformation can be defined by

$$h_{\nu,n}(f)(y) = \int_0^\infty (xy)^{\nu+1-(1/n)-(n\nu/2)} J_\nu^{(n)}(xy) f(x)\,dx\,,\ \text{for } y>0$$

with $\nu>-1+1/n$ and $n\in\mathbb{N}^*$. Note that $h_{\nu,n}$ reduces to the Hankel transformation with $n=2$. The $h_{\nu,n}$ transformation was introduced by M.T. Flores [1] who studied its main classical properties and investigated it on certain weighted L_p–spaces.

A.C. McBride [3] defined the space $F_{p,\mu}$ of functions as follows: if $1\leqslant p<\infty$ and $\mu\in\mathbb{R}$ a complex function $f\in C^\infty(0,\infty)$ is in $F_{p,\mu}$ if, and only if, $x^kD^k(x^{-\mu}f(x))\in L_\infty(0,\infty)$ for every $k\in\mathbb{N}$. If $p=\infty$ and $\mu\in\mathbb{R}$, $f\in C^\infty(0,\infty)$ is in $F_{p,\mu}$ provided that $x^kD^k(x^{-\mu}f(x))\longrightarrow 0$, as $x\longrightarrow\infty$ and $x\longrightarrow 0^+$, for every $k\in\mathbb{N}$. For $1\leqslant p\leqslant\infty$ and $\mu\in\mathbb{R}$, $F_{p,\mu}$ is endowed with the topology generated by the family of norms $\{\gamma_{p,\mu}^k\}_{k=0}^\infty$ where $\gamma_{p,\mu}^k(f)=\|x^kD^k(x^{-\mu}f(x))\|_p$ when $f\in F_{p,\mu}$ and $\|\cdot\|_p$ being the L_p–norm. The last author analized fractional integral operators [3,4] and the Hankel transform [5] on $F_{p,\mu}$.

In this note we study the $h_{\nu,n}$ transformation on the space $F_{p,\mu}$. Our results generalize some of the one due to A.C. McBride [5] for the Hankel transformation.

We firstly need to establish the following

PROPOSITION 1. *Let $f\in F_{p,\mu}$ with $p^{-1}-2-\nu+n^{-1}<\mu<p^{-1}$, $1\leqslant p\leqslant\infty$, $\nu>-1+n^{-1}$ and $n\in\mathbb{N}^*$. Then*

$$\delta^k h_{\nu,n}(f)(y) = (-1)^k h_{\nu,n}((\delta+1)^k f(x))(y),\ \text{for } y>0 \text{ and } k\in\mathbb{N},$$

where $\delta=yD$. Here and in the sequel p^{-1} is understood as 1 when $p=\infty$.

Proof. In virtue of the operational rule ([1, p.10])

$$D\left(x^{n(\nu+1)/2}J_{\nu+1}^{(n)}(x)\right) = n2^{-n/2}x^{-1+n(\nu+2)/2}J_\nu^{(n)}(x)$$

by integrating by parts we can obtain

$$h_{\nu,n}(f)(y) = -2^{n/2}n^{-1}\int_0^\infty (xy)^{1+\nu-(1/n)-n(\nu+1)/2}J_{\nu+1}^{(n)}(xy)\,(\delta+\eta)f(x)\,dx$$

where $\eta=2-(1+\nu)n+\nu-1/n$.

Differentiation under the integral sign leads to

$$\delta^k h_{\nu,n}(f)(y) = (-1)^{k-1} 2^{n/2} n^{-1} \int_0^\infty (xy)^{1+\nu-(1/n)-n(\nu+1)/2} J_{\nu+1}^{(n)}(xy)(\delta+1)^k(\delta+\eta) f(x)\, dx =$$

$$= (-1)^k h_{\nu,n}((\delta+1)^k f(x))(y)$$

for every $k \in \mathbb{N}$. ∎

The behaviour of the $h_{\nu,n}$ transformation is now showed.

THEOREM 1. *$h_{\nu,n}$ is a continuous operator from $F_{p,\mu}$ into $F_{p,(2/p)-1-\mu}$ provided that ν, n, μ and p satisfies the conditions in Proposition 1.*

Proof. By invoking the behaviours of $J_\nu^{(n)}$ near the origin and the infinity ([1, p.11]) we can write for $f \in F_{p,\mu}$

$$| y^{-\mu-1} h_{\nu,n}(f)(1/y) | \leqslant C\{I_{1,\nu+2-(1/n)+\mu,1}(|x^{-\mu} g(x)| + K_{1,-\mu,1}(|x^{-\mu} g(x)|)\}$$

where C is a suitable positive constant, $g(x) = (\delta+2-n(\nu+1)+\nu-1/n) f(x)$ and the operators I and K are defined by A. C. McBride in [3].

According to [3] and [7] we get

$$\| y^{\mu+1-(2/p)} \delta^k h_{\nu,n}(f)(y) \|_p \leqslant C_1 \| x^{-\mu}(\delta+1)^k(\delta+2-n(\nu+1)+\nu-1/n) f(x) \|_p.$$

Also one has

$$\| y^k D^k(y^{\mu+1-(2/p)} h_{\nu,n}(f)(y)) \|_p \leqslant C_2 \sum_{i=0}^m \gamma_{p,\mu}^i(f),$$

for $f \in F_{p,\mu}$ and certain $C_2 > 0$ and $m \in \mathbb{N}$. Thus the proof is finished. ∎

Let now $1 \leqslant p \leqslant \infty$, $\nu \in \mathbb{R}$, $n \in \mathbb{N}^*$ and $\mu > p^{-1}-2-\nu+n^{-1}$. We define the operator

$$h_{\nu,n}^*(f)(y) = (-2^{n/2} n^{-1})^k y^{-k} h_{\nu+k,n}(N_{\nu+k-1,n} \cdot N_{\nu+k-2,n} \cdots N_{\nu,n} f(x))(y), \quad f \in F_{p,\mu}$$

where $N_{\nu,n} = x^{-1}(\delta+\nu+2-n(\nu+1)-1/n)$ and $k \in \mathbb{N}$ is chosen such that $\nu+k > -1+1/n$ and $\mu < k+1/p$.

We can prove that $h_{\nu,n}^*$ is independent of k provided that k satisfies the above conditions. Moreover $h_{\nu,n}^* f = h_{\nu,n} f$, for $f \in F_{p,\mu}$ when $\nu > -1+1/n$ and $p^{-1}-2-\nu+n^{-1} < \mu < p^{-1}$. Hence $h_{\nu,n}^*$ can be seen as an extension of $h_{\nu,n}$ on $F_{p,\mu}$ to values of $\nu \leqslant -1+1/n$ and $\mu \geqslant p^{-1}$.

In virtue of Theorem 1 and [3, Theorem 2.6] we can establish the following

THEOREM 2. *If $1 \leqslant p \leqslant \infty$, $\nu \in \mathbb{R}$, $n \in \mathbb{N}^*$ and $\mu > p^{-1}-2-\nu+n^{-1}$, then $h_{\nu,n}^*$ is a continuous operator from $F_{p,\mu}$ into $F_{p,(2/p)-1-\mu}$.*

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