EXTRACTA MATHEMATICAE Vol. 6, Núms. 2 y 3, 96 – 114 (1991)

Some Problems for Suggested Thinking in Fréchet Space Theory

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AMS *Subject Class.* (1980): 46–02, 46A12, 46A05 *Received November* 8, 1991

CONTENTS

A mathematician friend describes himself by saying that he is not a mathematician in the sense of "one of those who proves theorems", but rather just a "maker of examples". This survey is in that spirit since it is intended basically to show a collection of examples. In doing so, we give an overall look at some problems arising in the study of the structure of Fréchet spaces which may deserve attention.

PART 1

1.1. A GENERAL PROBLEM: FRÉCHET SPACES AND ASSOCIATED BANACH SPACES. Let $E[\tau]$ be a locally convex space (l.c.s.), and denote by $\mathscr{U}(\tau)$ a fundamental system (f.s.) of absolutely convex closed neighborhoods of 0. Given a neighborhood $U \in \mathcal{U}(\tau)$ with associated seminorm p_U, let $\phi_U: E \longrightarrow E/\mathrm{Ker}\, p_U$ denote the quotient map. The completion of the normed space $E_U = (E/\mathrm{Ker}\, p_U, \|\cdot\|_U)$, where $\|\phi_U x\|_U = p_U(x)$ shall be referred to as the associated Banach space of U. It is thus clear that a locally convex topology τ on E implies the existence of a family $(\hat{E}_U)_{U \in \mathcal{U}(\tau)}$ of Banach spaces.

Could we reconstruct the space $E[\tau]$ starting with the family $(\hat{E}_U)_{U \in \mathcal{U}(\tau)}$? The answer is no: all locally convex spaces carrying the weak topology have associated Banach spaces isomorphic to some $\mathbb{K}^n$.

But this is the because we have forgotten an essential ingredient in our picture: if $V, U \in \mathcal{U}(\tau)$, $V \subseteq U$, then we have a natural linking–map $\hat{T}_{VU}$ between the associated Banach spaces which is the extension to the completion of the map $T_{VU}\phi_V x = \phi_U x$. What we now have is a family $(\hat{E}_U)_{U \in \mathcal{U}(\tau)}$ of Banach spaces and a family of $(\hat{T}_{VU})_{V,U \in \mathcal{U}(\tau)}$ of morphisms between them which has the connection property: $T_{WV}\, T_{VU} = T_{WU}$. The question is posed again: Could we reconstruct the space $E[\tau]$ starting with the families $(\hat{E}_U)_{U \in \mathcal{U}(\tau)}$ and $(\hat{T}_{VU})_{V,U \in \mathcal{U}(\tau)}$?

The answer is essentially yes.

The construction which passes from the families of spaces and maps to $E[\tau]$ is known as a projective limit. Projective limits of Banach spaces are complete, thus we shall obtain $E[\tau]$, if it is complete, or its completion if it is not.

But let us start again, this time with an arbitrary family $(X_i)_{i \in I}$ of Banach spaces and a family (T_{ji}) of maps having the connection property $(T_{jk}\, T_{ki} = T_{ji})$. Does there exist a locally convex space $E[\tau]$ having a fundamental system of neighborhoods of 0 $\mathcal{U}(\tau)$ with $(X_i)_{i \in I}$ as the family of associated Banach spaces and (T_{ji}) as the corresponding linking maps?

This seems to be an open problem.

In the Fréchet case, that is, when we restrict our attention to sequences of Banach spaces and of maps, an answer is found in a result of Dubinski:

THEOREM. ([21]) *Let $T_n: X_{n+1} \longrightarrow X_n$, $n = 1,2,\ldots$ be a sequence of linear continuous maps acting between Banach spaces, which are injective and have dense range. Then there exists a unique, up to isomorphisms, Fréchet space F having a fundamental sequence of norms for which the sequence of kinking maps is equivalent to (T_n).*

(Two operators $T: X \longrightarrow Y$ and $S: A \longrightarrow B$ are said to be equivalent if there exist isometries $I: X \longrightarrow A$ and $J: Y \longrightarrow B$ such that $T = J^{-1}SI$.)

We could be a little less exigent in our demands and simply ask if, given a fixed Banach space X, there is a locally convex space $E[\tau]$ having a fundamental system of associated Banach spaces isomorphic to X. The answer is yes, but the method for obtaining such an $E[\tau]$ is not clear:

THEOREM. ([44]) *Let B be an infinite-dimensional separable Banach space. Let $E[\tau]$ be a nuclear space. Then E has a f.s. of associated Banach spaces isometric to B.*

We see that this answer is very different in nature fromthat of Dubinski. It is time for a definition:

DEFINITION. Let $\mathscr{B}$ be a class of Banach spaces. Then define

$$\mathfrak{s}(\mathscr{B})=\{\text{l.c.s. } E[\tau] \text{ such that } \hat{E}_U \in \mathscr{B} \text{ for all } U \text{ of some } \mathscr{U}(\tau)\}.$$

Hopeless of resolution though it seems, this has to be our first problem:

PROBLEM 1. Characterize those l.c.s. in $\mathfrak{s}(\mathscr{B})$.

In practice one usually wants something more concrete, such as: Fréchet–Montel spaces in $\mathfrak{s}(c_0)$, Fréchet–Schwartz spaces in $\mathfrak{s}(L_p)$, Fréchet spaces having a f.s. of associated Banach spaces with the approximation property or the Dunford–Pettis property, and, in general, Fréchet spaces having some special property and a f.s. of associated Banach spaces of certain type. That will be our approach. An initial step is to see what happens if we focus our attention on the role of the linking maps instead of the associated Banach spaces.

1.2. ASSOCIATED BANACH SPACES AND GROTHENDIECK SPACE IDEALS. Let $\mathscr{A}$ be an operator ideal, that is a subclass of the class of all continuous operators acting between Banach spaces, which is closed under addition, and such that composition with continuous operators gives elements of $\mathscr{A}$. We assume, moreover, that the finite-dimensional operators are contained in $\mathscr{A}$. Two basic examples are $\mathscr{F}$, the finite-rank operators, and $\mathscr{L}$, all continuous operators.

Several interesting operator ideals result from considering those operators which transform a certain type of set into another type of set. For instance:

($\mathscr{K}$) Compact operators, those transforming bounded sets into relatively compact sets.

($\mathscr{W}$) Weakly compact operators, those transforming bounded sets into relatively compact sets.

As a special case, we have those operators defined by the property of sending sequences of one kind into sequences of another kind.

An intermission: Sequences in Fréchet spaces. A sequence (x_n) in a l.c.s. $E[\tau]$ is said to be...

...Banach–Saks if it has convergent arithmetic means.

...p–Banach–Saks, $1 < p < +\infty$, if, for any continuous seminorm Q, there is a constant $C(Q)$ such that

$$Q\left[\sum_{1\leqslant n\leqslant N} x_n\right] \leqslant C(Q)\, N^{1/p}.$$

...p–summable, $1 \leqslant p < +\infty$, if, for each continuous seminorm Q

$$\sum Q(x_n)^p < +\infty.$$

If $p = +\infty$, we obtain the τ–null sequences.

...weakly p–summable, $1 \leqslant p < +\infty$, if, for all $x^* \in E[\tau]^*$, $(x^* x_n) \in \ell_p$. Equivalently, if for each continuous seminorm Q there is a constant $C(Q) > 0$ such that

$$Q\left[\sum_k \xi_k x_k\right] \leqslant C(Q)\, \|(\xi_k)\|_{\ell_{p^*}}.$$

This last notion is nothing else than that of a p–summable sequence when $\tau = \sigma(E[\tau], E[\tau]^*)$. When $p = +\infty$, we simply obtain the weakly null sequences. We say that a sequence (x_n) is weakly–p–convergent (resp. p–Banach– Saks convergent) to x if $(x_n - x)$ is weakly–p–summable (resp. p–Banach– Saks).

Let K be a subset of a Fréchet space $E[\tau]$. We shall say that it is a relatively weakly–p–compact set if any bounded sequence admits a weakly–p–convergent sub–sequence. Notice that when $p = \infty$ we are simply speaking about weakly compact sets.

We now resume defining more operator ideals in Fréchet spaces:

(Π_p) p–Summing operators, those transforming weakly–p–summable sequences into p–summable ones.

$(\mathscr{U})$ Unconditionally converging operators, those sending weakly summable sequences into summable sequences.

$(\mathscr{B})$ Completely continuous or Dunford–Pettis operators, those sending weakly null sequences into norm–null sequences.

(C_p) p–converging operators, those sending weakly–p–summable sequences into null sequences. This scale of operators ideals is intermediate between $\mathscr{U} = C_1$ and $\mathscr{B} = C_\infty$.

(W_p) Weakly$-p-$compact operators, those transforming bounded sets into relatively weakly$-p-$compact sets.

A l.c.s. $E[\tau]$ which can be defined as a projective limit of Banach spaces with linking maps in $\mathscr{A}$ is said to be an $\mathscr{A}$-space. We also say that $E[\tau]$ belongs to the Grothendieck space ideal generated by $\mathscr{A}$, and we write this $E[\tau] \in \mathrm{Groth}(\mathscr{A})$.

Some of the most important classes of l.c.s., the so-called nuclear (N) and Schwartz (S) spaces are Grothendieck space ideals (see [27] for details): $N = \mathrm{Groth}(\Pi_1)$, and $S = \mathrm{Groth}(\mathscr{K})$. Other important classes, such as the Montel or barrelled spaces are not (see [11] for details).

Problems involving Grothendieck space ideals are numerous:

a) Determination problems.

b) Characterizations problems.

c) Uniqueness problems.

d) Extension problems.

Question a) is essentially: determine those classes $\mathfrak{A}$ of locally convex spaces such that, for some operator ideal $\mathscr{A}$, $\mathfrak{A} = \mathrm{Groth}(\mathscr{A})$; as we have already said, Montel or barrelled Schwartz spaces do not form a Grothendieck space ideal. This problem has been treated in [11].

Problems in b) are more difficult to fix precisely, since the word "characterizations" has several meanings. Perhaps the most obvious is: characterize, by means of a certain "inner" property, $\mathrm{Groth}(\mathscr{A})$ for concrete ideals $\mathscr{A}$. A great deal of literature deals with nuclear (and nuclear-type) or Schwartz spaces, and thus inner descriptions for them are well known (see [27]). Other cases are obvious: $\mathrm{Groth}(\mathscr{F})$ are the locally convex spaces carrying the weak topology. The choice $\mathscr{A} = \mathscr{G}$ (approximable operators = adherence of $\mathscr{F}$ in $\mathscr{L}$) has been treated in [8,9,10], and we shall give the available information in section 2.1.

The uniqueness problem could read: when is the operator ideal $\mathscr{A}$ which generates $\mathrm{Groth}(\mathscr{A})$ unique? A complete solution for nuclear-type ideals has been presented in [41]. When there is no uniquenes, the obtention of ideals $\mathscr{C}$

such that Groth($\mathscr{A}$) = Groth($\mathscr{C}$) seems to be an open problem.

Extension problems have been considered in [34].

The problem in which we are interested is the characterization of Banach spaces associated to $\mathscr{A}$–spaces, and therefore questions which could be classified b).

Nuclear and Schwartz spaces form the most interesting Grothendieck space ideals, and they are our starting point. The problem of Banach spaces associated to nuclear spaces is completely solved by Valdivia's result: they can be chosen isomorphic to any separable Banach space, i.e., for any separable Banach space X, $N \subseteq \mathfrak{s}(X)$. Not too much is known, however, about the structure of Banach spaces associated to Schwartz spaces (see [2]):

$$S = \mathfrak{s}(\text{subspaces of } c_0)$$
$$S \not\subseteq \mathfrak{s}(\text{subspaces of } \ell_p), \text{ for } 1 < p < +\infty.$$

The smaller subclass SH of Schwartz–Hilbert spaces, i.e., spaces in $S \cap \mathfrak{s}(\ell_2)$, has been studied in [3]. Fréchet spaces in this class are characterized by the following embedding property: if X is a Banach space and $F \in SH$, then F is a closed subspace of $X^{\mathbb{N}}$. Fréchet nuclear spaces also satisfy that property since Fréchet spaces in $\mathfrak{s}(X)$ are contained in $X^{\mathbb{N}}$.

The two classes are, however, different: if $\sigma \in c_0 \setminus \cup_{p>1} \ell_p$, $\sigma_n > 0$ for all n, then the projective limit

$$\longrightarrow \ell_2 \xrightarrow{D_\sigma} \ell_2 \xrightarrow{D_\sigma} \ell_2$$

defines a Schwartz–Hilbert non–nuclear space.

Conversely, a Fréchet space which belongs to $\mathfrak{s}(X)$ for any separable Banach space X must be nuclear; in fact we have the equation

$$N = \mathfrak{s}(c_0) \cap \mathfrak{s}(\ell_2)$$

whose proof is based on Grothendieck's inequality (see [27]). This raises a curious question:

QUESTION 1. Is there a Banach space X such that

$$N = S \cap \mathfrak{s}(X)?$$

(Unfortunatey c_0 or ℓ_2 do not serve for such an X: for example, the space (*), and the space obtained from it replacing ℓ_2 by c_0.)

Other elementary results, which we mention for the sake of completeness and later use, are:

$$\text{Groth}(\mathscr{W}) = \mathfrak{s}(\text{reflexive spaces})$$

(by the Davis–Figiel–Johnson–Pelczynski factorization method), and

$$\text{Groth}(\mathscr{F}) = \mathfrak{s}(\text{finite–dimensional spaces}).$$

It is clear that the membership of several classes $\mathfrak{s}(\mathscr{B})$ imposes restrictions on the structure of the space. For instance, if $p \neq q$,

$$\mathfrak{s}(\ell_p) \cap \mathfrak{s}(\ell_q) \subseteq S$$

by Pitt's lemma. Since almost everything in this area remains to be done, it is not difficult to pose intriguing questions. We would spotlight just three:

QUESTION 2. (L. Weiss) Let $\mathscr{S}$ denote the strictly singular (or Kato) operators. Characterize Fréchet spaces in Groth$(\mathscr{S})$.

QUESTION 3. Let $1 \leqslant p \leqslant +\infty$. Characterize Fréchet spaces in Groth(C_p).

QUESTION 4. Let $1 < p < +\infty$. Characterize Fréchet spaces in Groth$(\mathscr{W}_p)$.

PART 2

A second stage in the approach to problem 1 is to consider a property $(\mathscr{P})$ suitable to be possessed by Banach spaces (such as the approximation property, containing of c_0, reflexivity, etc). Imagine that we can define such a property $(\mathscr{P})$ for Fréchet spaces in a reasonable form (as is the case in the aforementioned examples).

PROBLEM 2. Are

1. $E[\tau]$ has property $(\mathscr{P})$.
2. $E[\tau] \in \mathfrak{s}$(Banach spaces with property $(\mathscr{P})$)

equivalent?

For instance, if $\mathscr{P}$ = reflexivity, 1 does not imply 2, while 2 implies 1.

If property $(\mathscr{P})$ can be described by means of an operator ideal $\mathscr{A}$ then we find again essentially two ways in which property $(\mathscr{P})$ can be defined in $E[\tau]$: requiring either that the identity of $E[\tau]$ be in $\mathscr{A}$, if a suitable extension of $\mathscr{A}$ is possible, or that $E[\tau] \in$ Groth$(\mathscr{A})$.

PROBLEM 3. Are

1. $E[\tau]$ has property $(\mathscr{P})$

2. $E[\tau] \in \mathfrak{s}$(Banach spaces with property $(\mathscr{P})$)
3. $E[\tau] \in \mathrm{Groth}(\mathscr{A})$
4. $\mathrm{id}(E[\tau]) \in \mathscr{A}$

equivalent?

Taking again our example $\mathscr{P}=$reflexivity $(\mathscr{A} = \mathscr{W})$, 2 and 3 are equivalent, and imply 1. Several possibilities are open to us for defining $\mathscr{W}$ in l.c.s. (see section 2.3). and the implications involving 4 will depend on that choice.

The rest of this section 2 will be devoted to considering these problems for the following choices of $(\mathscr{P})$:

AP: The approximation property and its variants.
P_0: X does not contain a copy of c_0.
P_1: X does not contain a copy of ℓ_1.
DP: The Dunford–Pettis property and its variants.
Properties of extraction of sub–sequences.

THE APPROXIMATION PROPERTY. Let us consider $\mathscr{P}=$aproximation property A.P., i.e., for any 0–nbhd $\mathscr{U}$ and any compact set K there is a finite rank operator T such that $(\mathrm{id}-T)(\mathrm{K}) \subseteq \mathscr{U}$. We shall also consider the bounded approximation property (B.A.P.), which, for a Fréchet separable space, means the existence of a sequence (T_n) of finite rank operators pointwise convergent to the identity. We shall denote by AP (resp. BAP) the class of Fréchet spaces having the approximation property (resp. the bounded approximation property).

Here, there is a question posed by Schottenloher [39]: Let F be a Fréchet space and let F_c^* denote its dual endowed with the topology of uniform convergence over the compact sets of F. We call F_c^* a DFC space. The question is:

QUESTION 5. (Schottenloher) Does every DFC space with A.P. belong to $\mathfrak{s}(BAP)$?

Let us denote by $\mathscr{G}$ the ideal of operators which are approximable in the operator norm by finite rank operators, i.e., $\mathscr{G}(X,Y)$ is the adherence of $\mathscr{L}\mathscr{F}(X,Y)$ in $\mathscr{L}(X,Y)$. Let us consider the Grothendieck space ideal $\mathrm{Groth}(\mathscr{G})$ generated by G. One can easily see that $N \subseteq \mathrm{Groth}(\mathscr{G}) \subseteq S$. Nuclear spaces have, in addition, the approximation property, as do $\mathscr{G}$–spaces; there are Fréchet

Schwartz spaces, however, without A.P. (see [26]). This suggests:

QUESTION 6. (Ramanujan) Is $S + \text{A.P.} = \text{Groth}(\mathscr{G})$?

Going back to Question 6, the following properties are equivalent [11,37]:

1) F has A.P., 2) $F_c^* \in \text{Groth}(\mathscr{G})$, 3) F_c^* has A.P.

Therefore, the question is whether a special subclass of $\mathscr{G}$−spaces is contained in $\mathfrak{s}(BAP)$. An attempt to clarify the structure of $\mathscr{G}$−spaces [11] led to a formulation of this problem in full generality:

QUESTION 7. Is $\text{Groth}(\mathscr{G}) = S \cap \mathfrak{s}(BAP)$?

To see to what extent there is an answer to this question, recall that a Fréchet nuclear space need not have B.A.P. [22]. A Schwartz space with B.A.P., however, must be a $\mathscr{G}$−space [8,37]. Thus, $\mathscr{G}$−spaces, which are intermediate between nuclear and Schwartz spaces, should have intermediate approximation properties. It turns out that [11]:

PROPOSITION. $E \in \text{Groth}(\mathscr{G})$ *if and only if it is a Schwartz space and has a "locally"* B.A.P.

That gives an inner characterization of $\mathscr{G}$−spaces. Moreover, it suggest an extension of Benndorf's results [1] for Schwartz spaces with B.A.P. (in turn an extension of a result of Pelczynski [36] for Banach spaces) to $\mathscr{G}$−spaces. This extension is shown to be possible by proving that [11]:

PROPOSITION. *$\mathscr{G}$−spaces are locally complemented subspaces of spaces in* $S \cap \mathfrak{s}(BAP)$.

(A subspace F of a l.c.s. $E[\tau]$ is said to be locally complemented if there is a f.s. of 0−neighborhoods in E, $\mathscr{U}(E)$, such that for any $U \in \mathscr{U}(E)$, the Banach space $\hat{F}_{U \cap F}$ is complemented in $\hat{E}_U$.)

Following a different line, Lourenço [31] improved the quality of the super−space by proving that:

PROPOSITION. *If E is a* DFC*−space with* A.P., *then E is a compact projective limit of a family of Banch spaces with a monotone Schauder basis.*

Although we do not know whether these results are sufficient to answer Questions 5 or 7, they seem to be enough to solve Schottenloher's original problem that lay behind Question 5 (see [32] for details).

2.1. CONTAINING OF COPIES OF c_0. It is well known that a Banach space X does not contain a copy of c_0 if and only if $\mathrm{id}(X) \in \mathscr{U}$ or, equivalently, $\mathrm{id}(X) \in C_1$. We shall abreviate $\mathrm{id}(X) \in C_1$ by $X \in \mathscr{C}_1$. What we want to know is whether

1. $E[\mathrm{t}]$ does not contain a copy of c_0
2. $E[\mathrm{t}] \in \mathfrak{s}(\mathscr{C}_1)$
3. $E[\mathrm{t}] \in \mathrm{Groth}(C_1)$

are equivalent.

It is easy to see that 1 and 3 are equivalent: this is a standar for Banach spaces (see [19]), and it can be extended to Fréchet spaces without further difficulties. But perhaps the most general result is due to Díaz Madrigal (see [20]). Recall first that a l.c.s. $E[\tau]$ is said to be Σ−complete if sequences in $\ell_1(E[\tau])$ define τ−summable series (notice that when $E[\tau]$ is Fréchet then $E[\tau]$ is Σ−complete if and only if $E[\tau] \in \mathscr{C}_1$). Then Díaz Madrigal's result is:

PROPOSITION. *Assume that* $E^*[\sigma(E^*,E)]$ *is* σ*−complete. Then* $E^*[\sigma(E^*,E^{**})]$ *is* σ*−complete if and only if* $E^*[\beta(E^*,E)]$ *does not contain a copy of* c_0.

Also, it is clear that $E[\tau] \in \mathrm{Groth}(C_1) \Longrightarrow E[\tau] \in \mathscr{C}_1$. Unfortunately, it can be shown that $E[\tau] \in \mathscr{C}_1$ does not imply $E[\tau] \in \mathrm{Groth}(C_1)$:

COUNTEREXAMPLE. (Köthe, Grothendieck, Valdivia) For any $1 \leqslant p < +\infty$, and $p = 0$ there exists a Fréchet Montel, non−Schwartz, échelon space of order p.

For $1 \leqslant p < +\infty$, these are the famous échelon spaces constructed by Köthe and Grothendieck [30,28]. The $p = 0$ the extension has been taken from Valdivia [45]. Let us consider this échelon space λ_0, which is a reduced projective limit of c_0. Because it is Montel it cannot contain a copy of c_0. But since $C_1(c_0,X) = \mathscr{K}(c_0,X)$ for any Banach space X, λ_0 cannot belong to $\mathrm{Groth}(C_1)$ unless it is a Schwartz space.

Thus the question arises:

QUESTION 8. Under which conditions are 2 and 3 equivalent?

We will mention here a conjecture of J.C. Díaz [17]. Recall that a l.c.s. $E[\tau]$

is said to be quasi–normable (see [23]) if for any equicontinuous subset A of E^* there is a 0–nbhd in E such that the topology induced on A by $E^*[\beta(E^*,E)]$ and $E^*_{V_0}$ coincide. The quasi–normability condition is receiving an increasing amount of attention (see for instance [4,5,6,7,35]).

CONJECTURE. If $E[\tau]$ is a quasi–normable space, then $E[\tau] \in \mathscr{C}_1$ if and only if $E[\tau] \in \text{Groth}(C_1)$.

And now let us present some arguments that support it:

2.2. CONTAINING OF COPIES OF ℓ_1. Köthe's λ_1 space shows that a Fréchet space can contain no copy of ℓ_1 and be, at the same time, a reduced projective limit of ℓ_1. This again provides a negative answer to problem P_1. Miñarro [33] shows that the answer to P_1 is positive for quasi–normable Fréchet spaces. This was also proved by M. Lindström (unpublished). The basic tool to handle the problem is Rosenthal's ℓ_1 theorem: a Banach space X does not contain a copy of ℓ_1 if and only if each bounded sequence admits a weakly Cauchy subsequence; i.e., if the unit ball is weakly conditionally compact. There is not too much difficulty in extending that characterization to Fréchet spaces [18], and thus we see that a Fréchet space $E[\tau]$ contains no copy of ℓ_1 if and only if bounded sets are weakly conditionally compact.

The second tool is the following lifting result of Miñarro [33]:

PROPOSITION. *Let $E[\tau]$ be a quasi–normable Fréchet space and let L be a closed subspace of E such that E/L is normable. Then the canonical quotient map $q: E \longrightarrow E/L$ lifts bounded sets (i.e., if B is a bounded set in E/L, there is a bounded set C in E such that $B \subseteq q(C)$).*

This, with some messy work, shows (see [33]):

PROPOSITION. *Let $E[\tau]$ be a quasi–normable Fréchet space. Then $E[\tau]$ has no copy of ℓ_1 if and only if it can be written as a projective limit of Banach spaces not containing ℓ_1.*

The difference between the cases ℓ_1 and c_0 is that in the former we know an equivalent formulation of the problem involving bounded sets; we do not know anything similar for c_0. Therefore it would be interesting to know how to:

QUESTION 9. (Díaz) Give a characterization of bounded sets in Banach spaces not containing a copy of c_0.

2.3. DUNFORD–PETTIS PROPERTIES. Introduced by Grothendieck in [24], the Dunford–Pettis property (in short D.P.) for a Banach space X is defined by any of the following equivalent conditions:

1. Weakly compact operators $T: X \longrightarrow Y$ transform relatively weakly compact sets into relatively compact sets.
2. Weakly compact operators $T: X \longrightarrow Y$ transform weakly convergent sequences into convergent sequences; i.e., $\mathscr{W}(X,Y) \subseteq \mathscr{B}(X,Y)$.
3. For any weakly null sequence (x_n) of X and any weakly nul sequence (x_n^*) of X^*, $\lim <x_n^*, x_n> = 0$.

Typical examples of Banach spaces having D.P. are $C(K)$ and $L_1(\mu)$ spaces. No reflexive Banach space can have D.P.

For an arbitrary l.c.s. $E[\tau]$, it is not clear that these properties are equivalent. It is not even clear what has to be understood by a weakly compact operator, since two definitions can be considered: operators transforming some 0–nhbd into a relatively weakly compact set, and operators transforming bounded sets into relatively weakly compact sets. We will call them, respectively, weakly compact ($\mathscr{W}$) and weakly bounded ($\mathscr{W}B$) operators. The following two definitions of the Dunford–Pettis property have been considered in the literature:

D.P. A l.c.s. $E[\tau]$ is said to have the Dunford–Pettis property if weakly bounded operators $T: E[\tau] \longrightarrow Y$, Y a Banach space, transform weakly compact sets into relatively compact sets.

s–DP. A l.c.s. $E[\tau]$ is said to have the strict Dunford–Pettis property if weakly bounded operators $T: E[\tau] \longrightarrow Y$, Y a Banach space, transform weakly convergent sequences into convergent sequences: i.e., $\mathscr{W}B(E[\tau], Y) \subseteq \mathscr{B}(E[\tau], Y)$.

We could consider two new possible definitions, replacing "weakly bounded" by "weakly compact". To this dazzling panorama we should add another definition:

seq–D.P. A l.c.s. $E[\tau]$ is said to have the sequential Dunford–Pettis property if, given weakly null sequences (x_n) and (x_n^*) in X and X^* respectively,

$$\lim <x_n^*, x_n> = 0.$$

In [29], Khurana proved that if $E[\tau]$ is quasi–complete then D.P.$\Rightarrow$s–D.P.

The converse is also true if $E^*[\sigma(E^*,E)]$ has a compact dense subset (see [29]). This leads to an example due to García and Gómez [25]:

COUNTEREXAMPLE. There is a quasi–complete l.c.s. having s–D.P. but not D.P.

The example is the space $E=\ell_\infty^*[\tau]$, where τ is the topology of uniform convergence over finite sequences of ℓ_∞ and over the set $\{e_n\}_n$; i.e., the topology having a subbase at zero the polars of the sets $S\subseteq\ell_\infty$ such that a) S is finite or b) $S=\{e_n\}_n$.

The space E has the Schur property, that is $\mathrm{id}(E)\in\mathscr{B}$. Therefore it has the seq–D.P. The space E has the strict DP but lacks the D.P., since the set $\{e_n\}_n$ is weakly relatively compact but not τ–relatively compact.

The sequential–D.P. has been examined in [16]. There it is proved that seq–D.P. implies strict D.P. for weakly compact operators (not for weakly bounded as in the results of Khurana and García–Gómez); also, that under an additional condition weaker than quasi–normability, the strict D.P. in the Mackey topology implies sequential D.P.

If we use D.P. to denote the class of Banach spaces having the Dunford–Pettis property, we can formulate another question:

QUESTION 10. Characterize Fréchet spaces in $\mathfrak{s}(DP)$.

The only plausible conjecture for this again comes from Díaz [17]:

CONJECTURE. Reflexive Fréchet spaces in $\mathfrak{s}(DP)$ are Montel.

It is not hard to verify that quasi–normable reflexive Fréchet spaces in $\mathfrak{s}(DP)$ are Montel. There exist FM spaces not in $\mathfrak{s}(DP)$ (see [23] for details).

2.4. EXTRACTION OF SUB-SEQUENCES. Let us consider the following general property defined for Fréchet or locally convex space:

Any sequence of type A contains a sub–sequence of type B.

In what follows we shall take for type A: "bounded sequence". Properties of this kind are:

Banach–Saks property: any bounded sequence admits a Banach–Saks sub–sequence.

p–Banach–Saks property, $1<p<+\infty$: any bounded sequence (x_n) admits a

p–Banach–Saks convergent sub–sequence.

W_p property, $1 \leqslant p < +\infty$: any bounded sequence admits a weakly–p–convergent sub–sequence. Notice that, in Fréchet spaces, W_∞ is exactly reflexivity.

If $\mathscr{P}$ is any of those properties, we denote by weak $\mathscr{P}$ the corresponding property obtained by replacing "bounded sequence" by "weakly null sequence". Weak properties are usually considered when the underlying space need not be reflexive. We shall use d_p to denote the weak W_p properties.

The study of this kind of properties in arbitrary l.c.s. spaces is terra incognita. It is clear that (weak) Banach–Saks property in l.c.s. spaces depends on the topology of $E[\tau]$, and not just upon the dual pair: all l.c.s. have, in the weak topology, the weak Banach–Saks property.

If we use BS to denote the class of Banach spaces having the Banach–Saks property, we see that Fréchet spaces in $\mathfrak{s}(BS)$ themselves have the BS property. Therefore:

QUESTION 11. Do Fréchet spaces in $\mathfrak{s}(BS)$ have the Banach–Saks property?

We conjecture that the answer to question 10 is negative.

It is obvious that $W_p \Rightarrow p^*$–Banach–Saks (and likewise for the weak version). Something which may seem surprising is that, for Fréchet spaces, the number

$$p_0 = \sup\{p : E[\tau] \text{ has the } p\text{–Banach–Saks property}\}$$

only depends upon the dual pair. This is because it can be proved that, in Banach spaces, p–Banach–Saks property implies W_r for all $r > p^*$ (see [15]). Obviously, a Fréchet space in Groth(W_p) itself has the property W_p, and these properties only depend upon the dual pair.

EXAMPLES. We write $X \in W_p$ instead of X has property W_p.

1. For $1 < p < +\infty$, $\ell_p[\sigma(\ell_p,\ell_{p^*})] \in W_{p^*}$.
2. For $1 < p < +\infty$, $L_p[\sigma(L_p,L_{p^*})] \in W_r$, where $r = \max\{2,p^*\}$.
3. Any co–nuclear space belongs to W_p, for all p.
4. Any Fréchet Montel space has the property W_1.

QUESTION 12. Is it true or false that $\ell_p[\sigma(\ell_p,\ell_r)] \in W_r$?

The implication $\mathrm{Groth}(W_p) \Rightarrow W_p$ helds only for Fréchet spaces: if J is uncountable, $\mathbb{K}^J$ is a simple counterexample to the converse implication. The space φ_d belongs to W_p but not to $\mathrm{Groth}(W_p)$. A Fréchet space $F \in W_1$ such that $F \notin \mathrm{Groth}(W_\infty)$ can be constructed: consider the Fréchet Montel, non–Schwartz, échelon space of order 0 constructed earlier. This space belongs to W_1 (by 4) and it cannot belong to $\mathrm{Groth}(W_\infty)$ since $\mathscr{W}(c_0,X) = \mathscr{K}(c_0,X)$.

Infinite–dimensional Banach spaces cannot have the W_1 property. Its weak version, the d_1 property, is, for a Banach space X, equivalent to the ∞–Banach–Saks property and to the so–called hereditary Dunford–Pettis property: any closed subspace of X has the Dunford–Pettis property. The situation for an arbitrary l.c.s. is, as we have just seen, somewhat different. It is still true that if $E[\tau]$ has property d_1 then it has the hereditary sequential Dunford–Pettis property, but the converse is false: the space $\mathbb{K}^I$, I uncountable, is a simple counterexample.

Still another related property is the following: a sequence (x_n) in a l.c.s. E is said to be very weakly convergent (briefly, r.w.c.) if, for some sequence (λ_n) of non–zero scalars, $(\lambda_n x_n)$ converges to 0 in E. A l.c.s. E is said to have property $\mathscr{C}'$ if every sequence in E contains a r.w.c. sub–sequence. Every Fréchet space has property $\mathscr{C}'$ and φ does not have it.

The following characterization is in [42].

PROPOSITION. *A l.c.s. E has property $\mathscr{C}'$ if and only if there is no infinite dimensional subspace F of E with the property that* $\dim \mathrm{B} \cap F < +\infty$ *for each bounded set* $\mathrm{B} \subset E$*; equivalently, if and only if for every subspace of E with countable dimension contains a bounded, absorbing set.*

QUESTION 13. If E has property $\mathscr{C}'$, is every sequence of E r.w.c.?

PART 3

3. EMBEDDING SUMS INTO PRODUCTS. A problem closely connected to the study of the associated Banach spaces that we should like to mention is the following: some l.c.s., such as (s) the universal nuclear space of rapidly decreasing sequences, have the property that the sum $\oplus_{\mathbb{N}}(s)$ can be embedded into large products $(s)^I$. For other spaces, like the finite–dimensional spaces, such an embedding is not possible. Thus we can pose the question of characterizing those l.c.s. $E[\tau]$ such that $\oplus_{\mathbb{N}} E[\tau]$ is a subspace of some product $E[\tau]^I$.

There are two meaningful extensions of this problem. One is to consider embeddings into different spaces: $\oplus_{\mathbb{N}} E[\tau] \longrightarrow F[\eta]^J$. Another to consider also the uncountable embedding problem: $\oplus_I E[\tau] \longrightarrow F[\eta]^J$.

When $E[\tau] = X$ is a Banach space, these problems have been treated in [43] (X a finite-dimensional Banach space, F any l.c.s.), [12] (case $X = H$ a Hilbert space), and [14] (X and F arbitrary Banach spaces). We shall briefly survey the main results and techniques encountered.

It is not hard to verify that the Banach spaces associated to $\oplus_{\mathbb{N}} X$ can be chosen isomorphic with $\ell_1^s(X)$, and that under that isomorphism the linking maps are diagonal operators

$$D_{\sigma^{-1}} : \ell_1^s(X) \longrightarrow \ell_1^s(X), \quad D_\sigma((x_n)) = (\sigma_n x_n).$$

Therefore, via a factorization argument, the associated Banach spaces can also be chosen isomorphic with $\ell_p^s(X)$, $0 < p < +\infty$, or $c_0^s(X)$, with diagonal linking maps:

$$\oplus_{\mathbb{N}} X = \varprojlim D_\sigma(\ell_p^s(X,I), \quad \sigma \in \ell_\infty^+(I)$$

(when $p = +\infty$, we understand $c_0^s(X,I)$). But we can, moreover, consider the topologies obtained by replacing ℓ_p^s by ℓ_p^w:

$$[\oplus_I X, \tau_p^w] := \varprojlim D_\sigma(\ell_p^w(X,I), \quad \sigma \in \ell_\infty^+(I)$$

(when $p = +\infty$, we understand $c_0^w(X,I)$). All these topologies, which are complete, coincide in the countable case, and satisfy

$$\tau_0 < \tau_{\text{box}} = \tau_\infty^w = \tau_\infty^s < \tau_p^w < \tau_p^s < \tau_1^s = \tau, \quad 1 \leqslant p \leqslant +\infty,$$

in the uncountable case (τ_0 and τ denote the product and the inductive topologies respectively). These topologies also have striking connections with tensor products:

$$[\oplus_I X, \tau_1^s] = [\varphi_d, \tau_1] \overline{\otimes}_\pi X,$$

$$[\oplus_I X, \tau_1^w] = [\varphi_d, \tau_1] \overline{\otimes}_\varepsilon X.$$

To extend the above lines to other τ_p-topologies we define the p-topologies $\tau(p)$ on the tensor product given by the seminorms

$$\tau_{p,\sigma}\left[\sum_{k=1}^N w_k \otimes x_k\right] = \left\| \left[\sum_{k=1}^N \sigma_k^{-1} w_{k,n} \otimes x_k\right]_n \right\|_{\ell_p^s(X)}$$

and thus intermediate between the ε and the π topologies on $\varphi_d \otimes X$. We have:

$$[\oplus_I X, \tau_p^s] = [\varphi_d, \tau_p] \overline{\otimes}_{\gamma(p)} X,$$

$$[\oplus_I X, \tau_p^w] = [\varphi_d, \tau_p] \overline{\otimes}_{\varepsilon} X.$$

Going back to the embedding $\oplus_I X \longrightarrow X^J$, let us mention the result for Hilbert spaces (see [12]):

PROPOSITION. *Let H be a Hilbert space. Then*

1) $[\oplus_I H, \tau_p^s]$ *is a subspace of* H^J *if and only if we have one of the following alternatives:*

a) $I = \mathbb{N}$, $\dim H = +\infty$ *and* $\operatorname{card} J \geqslant 2^{\aleph_0}$,

b) $p = 2$, $\dim H > d$ and $\operatorname{card} J \geqslant 2^d$.

2) $[\oplus_I H, \tau_p^w]$ *is a subspace of* H^J *if and only if* $I = \mathbb{N}$, $\dim H = +\infty$ *and* $\operatorname{card} J \geqslant 2^{\aleph_0}$.

We now turn our attention to the countable embedding $\oplus_{\mathbb{N}} X \longrightarrow X^I$. It can be shown that most of the natural Banach spaces (such as L_p or $C(K)$ spaces, vector sequence spaces, tensor products of these spaces, etc.) satisfy this embedding. Thus the question arises whether there is any infinite–dimensional Banach space not satisfying the countable embedding. The answer is yes: James' space, J, does not satisfy the countable embedding (see [14] for details).

Passing to arbitrary l.c.s., it would be nice to develop the corresponding theory for Fréchet spaces. It should be mentioned that the only non–trivial example of a Fréchet (non–Banach) space not satisfying the countable embedding is given by Simões [41], who constructs a nuclear $\Lambda_1(\alpha)$–space for which the embedding $\oplus_{\mathbb{N}} \Lambda_1(\alpha) \longrightarrow \Lambda_1(\alpha)^J$ is not possible.

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