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On Essentially Incomparable Banach Spaces

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ABSTRACT

We introduce the concept of essentially incomparable Banach spaces, and give some examples. Then, for two essentially incomparable Banach spaces X and Y, we prove that a complemented subspace of the product $X \times Y$ is isomorphic to the product of a complemented subspace of X and a complemented subspace of Y. If, additionally, X and Y are isomorphic to its respective hyperplanes, then the group of invertible operators in $X \times Y$ is not connected. The results can be applied to some classical Banach spaces.

Let X and Y be Banach spaces, and $L(X,Y)$ the class of all (continuous linear) operators from X to Y. Recall that $T \in L(X,Y)$ is Fredholm if it has finite dimensional null space $N(T)$ and finite codimensional range $R(T)$.

$S \in L(X,Y)$ is said to be inessential [8] if for any $V \in L(X,Y)$ we have that $I_X - VS$ is a Fredholm operator. We will denote

$$\mathfrak{I}(X,Y) := \{\, T \in L(X,Y) : \ T \text{ inessential} \}.$$

DEFINITION 1. We say that two Banach spaces X and Y are essentially incomparable if every operator from X to Y is inessential.

PROPOSITION 1. *The above definition is symmetric:*

$$L(X,Y) = \mathfrak{I}(X,Y) \iff L(Y,X) = \mathfrak{I}(Y,X).$$

Remark 1. *a)* If the dual spaces X^*, Y^* are essentially incomparable, then so are X and Y; but the converse is not true. There exists an hereditarily reflexive space Y whose dual is isomorphic to ℓ_1 [3].

b) X and Y are totally incomparable [9], [7, 2.c.1] (coincomparable [6]) if any space isomorphic to a subspace (quotient) of X and to a subspace (quotient) of Y is finite dimensional.

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Totally incomparable (coincomparable) Banach spaces are essentially incomparable.

Recall that an operator is completely continuous if it takes weakly compact sets into compact sets, and a Banach space Z has the Schur property if the identity operator in Z is completely continuous.

Z has the Dunford–Pettis property (DPP) if weakly compact operators from Z into any Banach space are completely continuous.

$C(K)$ spaces, L_1 spaces, the disc algebra A, the space of bounded analytic functions on the disc H^∞, Sobolev spaces, and other classical Banach spaces have the DPP (see [4]).

Z has the reciprocal Dunford–Pettis property (RDPP) if completely continuous operators from Z into any Banach space are weakly compact.

$C(K)$ spaces, as well as Banach spaces containing no copy of ℓ_1, or enjoying the property (V) of Pelczynski [5], have the RDPP.

THEOREM 1. *Y and Z are essentially incomparable in the following cases:*

a) Y reflexive and Z has the DPP.

b) Y has the RDPP *and Z has the Schur property.*

c) Y contains no copy of ℓ_1 and $Z=\ell_\infty$, H^∞, or $C(K)$, K σ–stonian.

d) Y contains no copy of c_0 and $Z=C(K)$.

e) Y contains no complemented copies of c_0 and $Z=C([0,1])$.

f) Y contains no complemented copies of ℓ_1 and $Z=L_1(\mu)$.

g) Y,Z different spaces from $\{\ell_p\ (1\leqslant p\leqslant\infty),\ c_0\}$.

PROPOSITION 2. *If Y and Z are essentially incomparable Banach spaces, then any Banach space isomorphic both to a complemented subspace of Y and to a complemented subspace of Z is finite dimensional.*

The analogy with the notions of totally incomparable [9] and totally coincomparable [6] Banach spaces suggests the following conjecture.

CONJECTURE. X and Y are essentially incomparable if and only if there exists no infinite dimensional Banach spaces isomorphic both to a complemented subspace of X and to a complemented subspace of Y.

Note that parts *c, d, e, f* and *g* in Theorem 1 support the conjecture, as well as the next theorem, proved in [1].

Recall [10] that a Banach space X is subprojective if every infinite dimensional subspace of X contains an infinite dimensional subspace complemented in X; and X is superprojective if every closed infinite codimensional subspace of X is contained in a closed infinite codimensional subspace complemented in X.

Examples of subprojective Banach spaces are c_0, ℓ_p $(1 \leqslant p < \infty)$, $L_p[0,1]$ $(2 \leqslant p < \infty)$, the separable hereditarily$-c_0$ spaces, and the dual of the original Tsirelson space; and as examples of superprojective spaces we have ℓ_p $(1 < p < \infty)$, $L_p[0,1]$ $(1 < p \leqslant 2)$, and the original Tsirelson space (see [1]).

THEOREM. ([1]) *Suppose X (or Y) is subprojective or superprojective. Then X and Y are essentially incomparable if and only if any space isomorphic to a complemented subspace of X and to a complemented subspace of Y is finite dimensional.*

COROLLARY 1. *a) $L_p[0,1]$ $(1 < p < \infty)$ is essentially incomparable with Z if and only if Z contains no complemented copies of ℓ_p or ℓ_2.*
b) ℓ_p $(1 \leqslant p \leqslant \infty)$ is essentially incomparable with Z if and only if Z contains no complemented copies of ℓ_p.
c) A separable hereditarily$-c_0$ space is essentially incomparable with Z if and only if Z contains no complemented copies of c_0.

THEOREM 2. *Suppose Y and Z are essentially incomparable. For every complemented subspace M of $Y \times Z$, there exists an invertible $U \in L(Y \times Z)$, and complemented subspaces Y_0 of Y and Z_0 of Z, so that $UM = Y_0 \times Z_0$.*

Remark 2. In [11] it is proved the result of theorem 2 for spaces X and Y such that any operator from X to Y is strictly singular, but there are essentially incomparable Banach spaces X and Y such that neither $L(X,Y)$ nor $L(Y,X)$ consist of strictly singular operators.

For example, $X := \ell_2(C[0,1])$ contains no complemented copies of ℓ_1 [2, Prop. 2.5]; hence it is essentially incomparable with $Y := L_1[0,1]$. However X contains a copy of ℓ_1 and a complemented copy of ℓ_2 and Y contains a complemented copy of ℓ_2 and a complemented copy of ℓ_1. Hence there are non$-$strictly singular operators from X into Y, and from Y into X.

COROLLARY 2. *Let $Y_1, \ldots, Y_n$ be pairwise essentially incomparable Banach spaces. Any complemented subspace of $Y_1 \times \cdots \times Y_n$ is isomorphic to a product $M_1 \times \cdots \times M_n$ with M_i a complemented subspace of Y_i.*

THEOREM 3. *Suppose Y and Z are essentially incomparable Banach spaces, both of them isomorphic to its respective hyperplanes. Then the group of all invertible operators in $Y \times Z$ is not connected.*

Remark. For different $X, Y \in \{\ell_p \ (1 \leqslant p \leqslant \infty),\ c_0\}$, we obtain a concrete example of an invertible operator in $X \times Y$ that cannot be connected with the identity, as follows:

$$T = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} : X \times Y \longrightarrow X \times Y$$

$A(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$, $D(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$, $C(x_1, x_2, \ldots) = (x_1, 0, 0, \ldots)$.

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