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A Note on the Lattice Definability of Bernstein Algebras [1]

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1. INTRODUCTION

The origin of Bernstein algebras lies in genetics and in the study of the stationary evolution operators, see Lyubich [10]. Holgate [7] was the first to give a formulation of Bernstein's problem into the language of nonassociative algebras. For a summary of known results see Wörz–Busekros [11, Ch. 9]. On the other hand, the study of the relationship between the lattice of subalgebras and other structural properties of the algebra can be found in Barnes ([3] and [2]) for associative and Lie algebras and J.A. Laliena for alternative algebras (see [8]). For Jordan algebras similar studies have been done by J.A. Laliena (see [9]) and completed by J.A. Anquela in [1].

Concerning Bernstein algebras, a first approach to the problem of determining the algebra by knowing its lattice of subalgebras can be found in [4], and in [6].

A finite–dimensional commutative algebra A over a field K is called baric if there exists a nontrivial homomorphism $\omega: A \longrightarrow K$, called the weight homomorphism.

A baric algebra is called a Bernstein algebra if:

$$(x^2)^2 = \omega(x)^2 x^2 \text{ for all } x \text{ in } A.$$

In the following, let K be a commutative infinite field of characteristic different from 2.

Let us list several results on Bernstein algebras which can be found in [11].

For every Bernstein algebra the nontrivial homomorphism $\omega: A \longrightarrow K$ is

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uniquely determined.

Every Bernstein algebra A posseses at least one non–zero indempotent.

Every Bernstein algebra A with non–zero idempotent e can be decomposed into the internal direct sum of subspaces (we will denote such a sum by $\dotplus$):

$$A = Ke \dotplus \operatorname{Ker}\omega, \quad \operatorname{Ker}\omega = U_e \dotplus V_e,$$

$$U_e = \{ex \,/\, x \in \operatorname{Ker}\omega\} = \{x \in A \,/\, ex = \tfrac{1}{2}x\}, \quad V_e = \{x \in A \,/\, ex = 0\}.$$

It can be proved that $U_e V_e \subseteq U_e$, $V_e^2 \subseteq U_e$, $U_e^2 \subseteq V_e$, $U_e V_e^2 = <0>$.

Although the decomposition of a Bernstein algebra depends on the choice of the idempotent e, the dimension of the subspace U_e of A is an invariant of A. If $\dim_K A = n+1$, then one can associate to $A = Ke \dotplus U_e \dotplus V_e$ a pair of integers $(r+1, s)$, called the type of A, where: $r = \dim_K U_e$, $s = \dim_K V_e$, so that $r+s = n$.

In the same way Wörz–Busekros shows in [11] that $\dim_K U_e^2$ and $\dim_K(U_e V_e + V_e^2)$ are also invariants of the algebra A. A Bernstein algebra is said to be trivial if $(\operatorname{Ker}\omega)^2 = 0$, exclusive if $U_e^2 = 0$ and normal if $U_e V_e + V_e^2 = 0$. Note that these definitions do not depend on the choice of the idempotent.

Let A be an algebra over a commutative field K. We denote by $\mathscr{L}(A)$ the lattice of all subalgebras of A. By an $\mathscr{L}$–isomorphism (lattice isomorphism) of the algebra A onto an algebra B over the same field, we mean an isomorphism:

$$\mathscr{L}(A) \longrightarrow \mathscr{L}(B)$$

of $\mathscr{L}(A)$ onto $\mathscr{L}(B)$.

We put $\ell(A)$, the length of A, for the supremum of the lengths of all chains in $\mathscr{L}(A)$ (by the length of a chain we mean its cardinality minus one). Clearly we have $\dim_K A \geqslant \ell(A)$ and if the algebra A is finite–dimensional then $\ell(\mathrm{A})$ is the maximum, not only the supremum. In [4], the following results are proved:

THEOREM 1.1. *For all subalgebras of a Bernstein algebra, length and dimension coincide. Hence the dimension of subalgebras is invariant by lattice isomorphism.*

THEOREM 1.2. *Let A, B be Bernstein algebras such that there is an $\mathscr{L}$–isomorphism between them. Then they must be isomorphic if some of the following conditions holds:*

i) Dimension of A is less than or equal to three.

ii) The algebra A is a trivial Bernstein algebra of any dimension and type.
iii) The algebra A is a normal Bernstein algebra of type $(2,n-1)$ $(\dim_K A = n+1)$.

2. Normal and Exclusive Bernstein Algebras

We begin with an example which shows that Theorem 1.2 cannot be extended to arbitrary Bernstein algebras.

EXAMPLE 2.1. Take the 4–dimensional Bernstein algebra listed as (5) in [5]:

$$A(\alpha) = Ke \dotplus Ku \dotplus Kv \dotplus Kw,$$

where $e^2 = e$, $eu = \frac{1}{2}u$, $ev = ew = 0$, $u^2 = 0$, $uv = 0$, $uw = 0$, $v^2 = u$, $w^2 = \alpha u$, $vw = 0$, $\alpha \notin -K^2$.

It is straighforward to see that the lattices of the $A(\alpha)$'s are isomorphic. In [5] it is shown that these algebras are not, in general, isomorphic. Nevertheless, although the algebra can not be completely determined, some structural characteristic can be known from the study of the lattice of subalgebras, as we see in the following.

THEOREM 2.2. *If two exclusive Bernstein algebras are $\mathscr{L}$–isomorphic, then they have the same type.*

THEOREM 2.3. *If two normal Bernstein algebras are $\mathscr{L}$–isomorphic, then they have the same type.*

References

1. ANQUELA, J.A., Lattice definability of semisimple Jordan algebras, *Communications in Algebra* **19**(5) (1991), 1409–1427.
2. BARNES, D.W., Lattice isomorphisms of Lie algebras, *J. Aust. Math. Soc.* **4** (1964), 470–475.
3. BARNES, D.W., Lattice isomorphism of associative algebras, *J. Aust. Math. Soc.* **6** (1966), 106–121.
4. CORTÉS, T., Bernstein algebras: lattice isomorphisms and isomorphisms, to appear in the "Proceedings of Jornadas sobre Modelos Algebraicos no Asociativos y sus Aplicaciones (April 1989)", Univ. Zaragoza, Nova Science Publishers, Inc.
5. CORTÉS, T., Classification of 4–dimensional Bernstein algebras, *Communications in Algebra* **19**(5) (1991), 1429–1443.
6. GONZÁLEZ, S. AND MARTÍNEZ, C., Idempotent elements in a Bernstein algebra, *J. London Math. Soc.* **42** (1990), 430–436.
7. HOLGATE, P., Genetic algebras satisfying Bernstein's stationarity principle, *J. London Math. Soc.* **9** (1975), 613–623.

8. LALIENA, J.A., Lattice isomorphisms of alternative algebras, *J. Alg.* **128** (1990), 335 – 355.
9. LALIENA, J.A., Lattice isomorphisms of Jordan algebras, to appear in the "Proceedings of Jornadas sobre Modelos Algebraicos no Asociativos y sus Aplicaciones (April 1989)", Univ. Zaragoza, Nova Science Publishers, Inc.
10. LYUBICH, YU L., Basic concepts and theorems of the evolutionary genetics of free populations, *Russian Math. Surveys* **26** (1971), 51 – 123.
11. WÖRZ-BUSEKROS, A., "Algebras in Genetics", Lecture Notes in Biomathematics Vol. 36, Springer–Verlag, Berlin Heidelberg, 1980.