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Some Results on the Duals of the Inductive and Projective Limits of Moscatelli Type

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From now on, $(L,\|\cdot\|)$ will denote a normal Banach sequence space i.e. a Banach sequence space satisfying:

(α) $\varphi \subset L \subset \omega$ algebraically and the inclusion $(L,\|\cdot\|) \longrightarrow \omega$ is continuous (here ω and φ stands for $\mathbb{K}^{\mathbb{N}} = \Pi_{k\in\mathbb{N}} \mathbb{K}$ and $\oplus_{k\in\mathbb{N}} \mathbb{K}$ respectively).

(β) $\forall a=(a_k)\in L, \forall b=(b_k)\in\omega$ with $|b_k| \leqslant |a_k|$ $(k\in\mathbb{N})$, we have $b\in L$ and $\|b\| \leqslant \|a_k\|$.

Clearly, every projection onto the first n coordinates $p_n: \omega \longrightarrow \omega$, $(a_k)_{k\in\mathbb{N}} \longrightarrow ((a_k)_{k<n},(0)_{k>n})$ induces a norm–decreasing endomorphism on L.

Other properties on $(L,\|\cdot\|)$ we may require are the following:

(γ) $\|a\| = \lim \|p_n(a)\|$ $(n \to \infty)$, $\forall a\in L$.

(ϵ) $\lim \|a - p_n(a)\| = 0$ $(n \to \infty)$, $\forall a\in L$ (i.e. φ is dense in $(L,\|\cdot\|)$).

(δ) If $a\in\omega$, and $\sup_{n\in\mathbb{N}} \|p_n(a)\| < \infty$, then $a\in L$ and $\|a\| = \lim \|p_n(a)\|$ $(n \to \infty)$.

For a locally convex space Z, we shall denote by $\mathfrak{b}(Z)$ the set of all bounded sets in Z and by Z'_b its strong dual.

Unexplained terminology as in [6,7,11].

DEFINITION. Let $(L,\|\cdot\|)$ be a normal Banach sequence space, let Y and X be locally convex spaces and $f: Y \longrightarrow X$ a continuous linear mapping. For every $n\in\mathbb{N}$, we define $F_n := \Pi_{k<n} Y \times L((X)_{k\geqslant n})$ provided with the topology of such a finite topological product. For every $n\in\mathbb{N}$, we also define the mapping $g_n: F_{n+1} \longrightarrow F_n$, $(x_k)_{k\in\mathbb{N}} \longrightarrow ((x_k)_{k<n}, f(x_n), (x_k)_{k>n})$. Clearly g_n $(n\in\mathbb{N})$ is a continuous linear mapping. We define the projective limit F of Moscatelli type w.r.t. (with respect to) $(L,\|\cdot\|)$, Y, X and $f: Y \longrightarrow X$ by $F = \mathrm{proj}_{n\in\mathbb{N}} (F_n, g_n)$.

PROPOSITION 1. *Let* $(L,\|\cdot\|)$ *be a normal Banach sequence space which fulfils property* (ϵ) *and let* Z *be a locally convex space such that*

i) For every $\mathfrak{B}\in\mathfrak{b}(L(Z))$, *there is* $B\in\mathfrak{b}(Z)$ *with* $\mathfrak{B}\in\mathfrak{b}(L(Z_B))$.

ii) For every $u\in L'(Z'_b)$, *there is an absolutely convex equicontinuous set* $M\subset Z'$ *with* $u\in L'(Z'_M)$.

Then $L(Z)'_b$ *is canonically algebraically and topologically isomorphic to* $L'(Z'_b)$.

Remark. In particular *i)* and *ii)* are satisfied if either L satisfies (δ) and Z is a quasi–barrelled DF–space or L satisfies (ϵ) and Z is metrizable.

In order to establish now the duality between the general inductive and projective limits of Moscatelli type, let us first recall the definition of the inductive ones.

Let $(L,\|\cdot\|)$ be a normal Banach sequence space, let Y and X be locally convex spaces, Y continuously included in X. For every $n\in\mathbb{N}$, the space $E_n:=\Pi_{k<n}X\times L((Y)_{k\geqslant n})$ has the obvious meaning and should be provided with the canonical product topology. Now we define the inductive limit of Moscatelli type w.r.t. $(L,\|\cdot\|),X,Y$ (and the continuous canonical inclusion $j\colon Y\longrightarrow X$) as $E=\text{ind}_{n\in\mathbb{N}}E_n$ (we refer to [8] for details).

PROPOSITION 2. *Let* $(L,\|\cdot\|)$ *be a normal Banach sequence space with property* (ϵ), *let* Y *and* X *be a locally convex spaces and* $f\colon Y\longrightarrow X$ *a continuous linear mapping with dense range. Let* F *be the corresponding projective limit of Moscatelli type. Let* E *be the inductive limit of Moscatelli type w.r.t. the duals* $(L',\|\cdot\|'),X',Y'_b$ *and* $f^t\colon X'_b\longrightarrow Y'_b$ *(that we shall always consider as an inclusion). If the following two conditions are satisfied:*

i) For every $\mathfrak{B}\in\mathfrak{b}(L(X))$ *there is* $B\in\mathfrak{b}(X)$ *such that* $\mathfrak{B}\in\mathfrak{b}(L(X_B))$.

ii) For every $u\in L'(X'_b)$, *there is an absolutely convex* X*–equicontinuous set* $M\subset X'$ *such that* $u\in L'(X'_M)$.

Then $F'=E$ *algebraically and* E *is continuously embedded in* F'_b.

The topological identity in the proposition above is rather delicate. We refer to [3] and [5] for the case of Banach spaces Y and X.

PROPOSITION 3. *Let* $(L,\|\cdot\|)$ *be a normal Banach sequence space with property* (ϵ), *let* Y *and* X *be a locally convex spaces,* Y *continuously included in* X. *Let* E *be the inductive limit of Moscatelli type w.r.t.* $(L,\|\cdot\|)$, Y,X *(and the* *continuous canonical inclusion* $j: Y \longrightarrow X$*). Let* F *be the corresponding projective limit of Moscatelli type w.r.t. the duals* $(L', \|\cdot\|'), X'_b, Y'_b$ *and* j^t*. If the following two conditions are satisfied:*

i) For every $\mathfrak{B} \in \mathfrak{b}(L(Y))$ *there is* $B \in \mathfrak{b}(Y)$ *such that* $\mathfrak{B} \in \mathfrak{b}(L(Y_B))$*.*

ii) For every $u \in L'(Y'_b)$*, there is an absolutely convex* Y*−equicontinuous set* $M \subset Y'$ *such that* $u \in L'(Y'_M)$*.*

Then $E' = F$ *algebraically, the inclusion* $E'_b \subset F$ *is continuous and* $E'_b = F$ *algebraically and topologically whenever* E *is regular.*

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