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Some Classes of Linearly Topologized Spaces

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Given a Tychonoff space X, let $C_c(X)$ be the ring of all real valued continuous functions defined on X endowed with the locally convex topology of uniform convergence on compact subsets of X. Let us recall that a subset A of X is topologically bounded if $f(A)$ is bounded for each $f \in C(X)$. X is said to be a μ–space if each topologically bounded subset of X is relatively compact. X is said to be replete if X coincides with its realcompactification νX. The classical results of Nachbin and Shirota ([4],[12] and [6]), which allowed to obtain barrelled spaces that are not bornological state:

THEOREM 1. *$C_c(X)$ is barrelled if and only if X is a μ–space.*

THEOREM 2. *$C_c(X)$ is bornological if and only if X is replete.*

Later on, De Wilde and Schmets [2] proved the latter to be true iff $C_c(X)$ is ultrabornological. And the theorems of Buchwalter and Schmets ([1] and [5]) consider the space $C_s(X)$ obtained by endowing $C(X)$ with the locally convex topology of pointwise convergence on X:

THEOREM 3. *$C_s(X)$ is barrelled if and only if each bounding subset of X is finite.*

THEOREM 4. *$C_s(X)$ is bornological if and only if X is replete.*

THEOREM 5. *$C_s(X)$ is ultrabornological if and only if X is replete and each compact subset of X is finite.*

In this note we present two different topologies on $C(X)$ which allow to obtain similar results in the realm of linearly topologized spaces. Let us recall a linearly topologized space [3, §10] is a Hausdorff topological vector space L over a discrete field K that has a base of neighbourhoods of the origin $\mathcal{U}$ consisting of linear subspaces, and a base of neighbourhoods of each $x \in L$ is obtained by taken all the $x+U$, $U\in \mathcal{U}$.

Given $f\in C(X)$, let f^* be the continuous extension of f to the Stone–Čech compactification βX of X which takes values in the Alexandroff compactification of $\mathbb{R}$, and as usual $\operatorname{supp} f$ (resp., $\operatorname{supp} f^*$) will denote the support of f (resp., of f^*). By [11, II.1.3], for each linear subspace H of $C(X)$ there is a minimum compact subset sH of βX such that if $sH\cap \operatorname{supp} f^* = \emptyset$, then $f\in H$.

Let us consider the linearly topologized space $C_\lambda(X)$ (resp., $C_\sigma(X)$) obtained by endowing $C(X)$ with the topology that admits as a base of neighbourhoods of the origin the linear subspaces $L_K := \{f\in C(X) : K\cap \operatorname{supp} f = \emptyset\}$, for each compact (resp., finite) subset K of X, and defined over the discrete field $\mathbb{R}$. Then the following two characterizations of the linear subspaces of $C_\lambda(X)$ and $C_\sigma(X)$ hold, [8, Proposition 4] and [7, Lemma 1],

PROPOSITION 1. *A linear subspace H of $C_\lambda(X)$ is open if and only if $sH\subset X$.*

PROPOSITION 2. *A linear subspace H of $C_\sigma(X)$ is open if and only if sH is a finite subset of X.*

A linear subspace F of a linearly topologized space L is linearly bounded [3, §13.1] if $\dim(F+U)/U$ is finite for each linear neighbourhood U of the origin. Let $\mu(L',L)$ be the finest linear topology on the topological dual L' of L that has L as dual space, then L is endowed with the linear strong topology if a base of neighbourhoods of the origin in L is formed by the orthogonal spaces to the $\mu(L',L)$–bounded subspaces of L'. The classes of spaces considered in order to obtain the equivalent results to the theorems of Nachbin–Shirota, De Wilde–Schmets and Buchwalter–Schmets in the context of linearly topologized spaces are the following:

DEFINITION. A linearly topologized space L is called:

– linearly barrelled if L is endowed with the linear strong topology.

– linearly bornological if given any linear mapping T from L to any linear topologized space F, then T is continuous if the restriction of T to each linearly bounded metrizable subspace of L is continuous.

– linearly ultrabornological if given any linear mapping T from L to any linearly topologized space F, then T is continuous if the restriction of T to each subspace of L that is isomorphic to the topological product of a countable infinity of copies of K is continuous.

Other equivalent definitions may be found in [8–10]. All these classes of spaces enjoy good permanence properties, e.g. they are stable under linear inductive limits, topological direct sums and quotients by closed subspaces; completions and topological product of linearly barrelled spaces are barrelled; (closed) subspaces and countable products of linearly (ultra)bornological spaces are linearly (ultra)bornological; and, if d is a non countable cardinal, the topological product of d linearly (ultra)bornological spaces is (ultra)bornological if and only if the topological product $\omega_d(K)$ is (ultra)bornological. The results obtained are:

THEOREM 6. *$C_\lambda(X)$ is linearly barrelled if and only if X is a μ−space.*

THEOREM 7. *$C_\lambda(X)$ is linearly bornological (ultrabornological) if and only if X is replete.*

THEOREM 8. *$C_\sigma(X)$ is linearly barrelled if and only if any bounding subset of X finite.*

THEOREM 9. *$C_\sigma(X)$ is linearly bornological if and only if X is replete.*

THEOREM 10. *$C_\sigma(X)$ is linearly ultrabornological if and only if X is replete and each compact subset of X is finite.*

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