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The Global Control of Nonlinear Partial Differential Equations and Variational Inequalities

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We study in this note the control of nonlinear diffusion equations and of parabolic variational inequalities by means of an approach which has been proved useful in the analysis of the control of nonlinear ordinary differential equations ([3]) and linear partial differential equations ([2] and [3]). It is based on an idea of Young [7], consisting in the replacement of classical variational problems by problems in measure spaces; its extension to optimal control problems, and the realization that one is dealing with fully linear problems – even if the original problems were nonlinear in the usual sense – is due to us [3]; see also the review paper [4] for a full bibliography as well as a historical analysis of these matters.

We shall consider a nonlinear difussion equation and a parabolic variational inequality, with boundary controls; we wish to minimize integral perfomance criteria, given that the terminal state should be fixed. We first write some well-known integral relationships satisfied by the solution of the equations, and then proceed to transform the problems; instead of minimizing over a set of admissible pairs trajectory-control, we find that it is possible to minimize over a product of two measure spaces. The advantages of the new formulation are: (i) an automatic existence theory – there always is a minimizer for our measure-theoretical problem; (ii) the new problem is linear, and then one can use the whole paraphernalia of linear analysis for dealing with such a problem; (iii) the minimization is *global* – the value reached, say numerically, is close to what one could reasonably call the global infimun of the problem. The prize to pay for these advantages is that the final state is reached only asymptotically – that is, as the number of (linear) constraints associated with the measure-theoretical problem tends to infinity; the situation is similar to our results in the finite-dimensional case; see [3], Chapter 4. A computational method has been developed, in which we treat the semi-infinite linear programming problems thus developed by means of simplex methods; nearly-optimal controls can be constructed in this manner.

DESCRIPTION

Let D be a bounded domain in $\mathbb{R}^n$ with smooth boundary ∂D, and T a positive real number, and define $Q_T:=D\times(0,T)$, $\Gamma_T:=\partial D\times(0,T)$, $D_T:=D\times\{T\}$. We shall consider two types of systems; the first, a nonlinear diffusion equation,

$$u_t(x,t)-\operatorname{div}(k(x)\nabla u(x,t))=f(u(x,t),x,t), \tag{1}$$

for $(x,t)\in Q_T$, with the initial condition $u(x,0)=0$, $x\in D$, and the boundary condition $\nabla u\cdot n|_{\Gamma_T}=v$; n is the outward normal, and the function $(s,t)\in\Gamma_T\longrightarrow v(s,t)\in V\subset\mathbb{R}$ is the *control function*, taking values in a *bounded* control set V. The terminal relationship $u(\cdot,T)=g$ is to be satisfied; g is a given continuous function on D_T. We transform now this problem, with a view at generalization. Let φ be in a subset of $C^2(\overline{Q}_T)$, which we shall call $K^2(\overline{Q}_T)$, and which consists of those functions in $C^2(\overline{Q}_T)$ whose normal derivative on Γ_T is zero. Then

$$\int_{Q_T}[u(\varphi_t+k\Delta\varphi+\nabla\varphi\nabla k)+f\varphi]\,dxdt=-\int_{\Gamma_T}k\varphi v\,dsdt+\int_{D_T}g\varphi\,dx, \tag{2}$$

for all $\varphi\in K^2(\overline{Q}_T)$. Since the control set V is bounded, then there is a *bounded* set $A\subset\mathbb{R}$ so that $u(x,t)\in A$, $\forall(x,t)\in\overline{Q}_T$.

Our second type of system is described by a variational inequality. Consider the following spaces: $H:=L^2(D)$, $V:=H^1(D)$, $W(0,T):=\{v: v\in L^2(0,T;V),\ v'\in L^2(0,T;V')\}$, $W_0(0,T):=\{v: v\in W(0,T),\ v(0)=u_0,\ u_0 \text{ given in } H\}$, and the (continuous, but not differentiable) function ϕ, defined by:

$$\phi(\lambda):=\begin{cases} g_1(\lambda-h_1) & \lambda\leqslant h_1\\ 0 & h_1\leqslant\lambda\leqslant h_2\\ g_2(\lambda-h_2) & h_2\leqslant\lambda\end{cases}$$

with $g_1<0<g_2$, $h_1\leqslant h_2$. We can consider the variational inequality [1]:

$$\int_{Q_T}[u_t(\varphi-u)+\nabla u\nabla(\varphi-u)-f(\varphi-u)]\,dxdt+\int_{\Gamma_T}[\phi(\varphi)-\phi(u)]\,dsdt\geqslant 0,$$
$$u\in W_0(0,T),\ \forall\varphi\in W(0,T). \tag{3}$$

We shall then call v the control function $u|_{\Gamma_T}$, so that our boundary condition is $u|_{\Gamma_T}=v$; the control function takes values in a *bounded* set V'. We shall also transform this problem. We shall choose $u_0=0$ in the definition of $W_0(0,T)$, and put $\varphi-u=w$; then

$$\int_{Q_T}[u(-w_t-\Delta w)-fw]\,\mathrm{d}x\mathrm{d}t+\int_{D_T}gw\,\mathrm{d}x+ \tag{4}$$

$$+\int_{\Gamma_T}[\phi(w+v)-\phi(v)-v\nabla w\cdot n]\,\mathrm{d}s\mathrm{d}t\geqslant 0\,,\quad \forall\, w\in W(0,T).$$

Again, since the control set V' is bounded, then there is a *bounded* set $A'\subset\mathbb{R}$ so that $u(t,x)\in A'$ $\forall\,(x,t)\in\overline{Q}_T$. The terminal relationship $u(\cdot,T)=g$ is also to be satisfied; g is, again, a given continuous function on D_T.

We shall consider two problems. Problem P consists of the minimization of the integral functional

$$J(u,v):=\int_{Q_T}f_0(u(x,t),x,t)\,\mathrm{d}x\mathrm{d}t+\int_{\Gamma_T}f_1(v(s,t),s,t)\,\mathrm{d}s\mathrm{d}t\,, \tag{5}$$

over the set of admissible pairs for the first of our systems; the second, Problem P', of the same functional for the second system, the variational inequality. The functions f_0 and f_1 are continuous functions defined in the appropriate spaces.

METAMORPHOSIS

In general, the minimization of the functional (5) over the set of admissible pairs for problems P or P' is not possible – the infimum is not attained at any admissible pair. We proceed then to transform these problems, realizing that a solution of (2) or of (4) defines a linear, bounded positive functional $u(\cdot,\cdot):F\longrightarrow \int_{Q_T}F(u(x,t),x,t)\mathrm{d}x\mathrm{d}t$ in the space $C(\Omega)$ of continuous real-valued functions F, with $\Omega:=A\times Q_T$ for problem P, and $\Omega:=A'\times Q_T$ for P'. Also, a control v defines a linear, bounded, positive functional: $v(\cdot,\cdot):G\longrightarrow \int_{\Gamma_T}G(v(s,t),s,t)\mathrm{d}s\mathrm{d}t$ in the space $C(\omega)$ of continuous functions G, $\omega:=V\times\Gamma_T$ for P, and $\omega:=V'\times\Gamma_T$ for P'. By Riez's theorem, an admissible pair (u,v) defines two Radon measures μ and ν, the first on Ω, the second on ω, so that (2) becomes:

$$\int_\Omega F_\varphi\,\mathrm{d}\mu+\int_\omega G_\varphi\,\mathrm{d}\nu=\int_{D_T}g\varphi\,\mathrm{d}x:=\alpha_\varphi\,,\quad \forall\,\varphi\in K^2(\overline{Q}_T), \tag{6}$$

where

$$F_\varphi(u,x,t):=u[\varphi_t(x,t)+k(x,t)\Delta\varphi(x,t)+\nabla k(x,t)\nabla\varphi(x,t)]+f(u,x,t)\varphi(x,t),$$

$$G_\varphi(v,s,t):=k\varphi(s,t)v.$$

Also, for problem P', (4) becomes

$$\int_\Omega F'_w\,\mathrm{d}\mu+\int_\omega G'_w\,\mathrm{d}\nu\leqslant\int_{D_T}gw\,\mathrm{d}x:=\alpha'_w\,,\quad \forall\,w\in W(0,T), \tag{7}$$

where

$$F'_w(u,x,t) := u[w_t(x,t) + \Delta w(x,t)] + f(u,x,t)w(x,t),$$

$$G'_w(v,s,t) := -\phi(w(x,t)+v) + \phi(v) + v\nabla w(s,t)\cdot n.$$

Thus, the minimization of the functional (5) over the set of admisible pairs is equivalent to the minimization of

$$I(\mu,\nu) = \mu(f_0) + \nu(f_1), \tag{8}$$

where we have written $\mu(f)$ for $\int_\Omega f\,d\mu$, and $\nu(g)$ for $\int_w g\,d\nu$, over the set of measures (μ,ν) corresponding to admissible pairs, which satisfy

$$\mu(F_\varphi) + \nu(G_\varphi) = \alpha_\varphi, \quad \forall\,\varphi\in K^2(\overline{Q}_T), \tag{9}$$

in the case of problem P, and

$$\mu(F'_w) + \nu(G'_w) \leqslant \alpha'_w, \quad \forall\, w\in W(0,T), \tag{10}$$

for problem P'. So far, we have not achieved anything new. We consider the extension of our problems; we shall consider the minimization of (8) over the set S of all pairs of measures (μ,ν) in $\mathcal{M}^+(\Omega)\times\mathcal{M}^+(\omega)$ satisfying (9) for problem P and (10) for P', plus the extra condition, satisfied of course by the admissible pairs, that these measures project on the (x,t) or (s,t) plane as the respective Lebesgue measures. (See [3] for a further discusion of this point.) Thus, if a function $\xi:\Omega\longrightarrow\mathbb{R}$ depends only on (x,t), then $\mu(\xi)=\alpha_\xi$, the Lebesgue integral of ξ over Q_T. Also, if a function $\zeta:\omega\longrightarrow\mathbb{R}$ depends only on (s,t), then $\nu(\zeta)=b_\zeta$, the Lebesgue integral of ζ over Γ_T. Finally, we have, writting 1_Ω and 1_ω for the characteristic functions of Ω and ω, $\mu(1_\Omega)\leqslant T$, $\nu(1_\omega)\leqslant T$; we have assumed that the Lebesgue measures of D and ∂D equal unity.

EXISTENCE, APPROXIMATION AND COMPUTATION

The proofs of the following Propositions are much like that of Theorem II.1 in [3] and the Appendix in [2] respectively, for both cases P and P':

PROPOSITION 1. *There exists an optimal pair* $(\mu^*,\nu^*)\in S$ *which minimizes the functional* I.

PROPOSITION 2. *The set* $S_1\subset S$ *of measures* (u,v) *which are piecewise--constant functions on* Ω *and* ω *respectively and satisfy* (9) *or* (10) *and the rest of the constraints is weakly*-dense in* S.

Let $\{\varphi_i : i=1,2,...\}$ be a set of functions which is *total* in $K^2(\overline{Q}_T)$, that is, whose linear combinations are uniformly dense in this space; we shall write $F_i := F_{\varphi_i}$, $G_i := G_{\varphi_i}$, $\alpha_i := \alpha_{\varphi_i}$, $\forall i$. Let further $\{w_i : i=1,2,...\}$ be a set of functions which is total in $W(0,T)$. We shall write $F_i' := F_{w_i}'$, $G_i' := G_{w_i}'$, $\alpha_i' := \alpha_{w_i}'$, $\forall i$. Further, we shall also take sets of functions $\{\xi_j : j=1,2,...\}$ and $\{\zeta_k : k=1,2,...\}$ which are total in the respective subspaces of $C(\Omega)$ and $C(\omega)$, writing a_j for a_{ξ_j} and b_k for b_{ζ_k}. We have then our main result of approximation; the proof is much like that of Proposition III.1 in [3]:

PROPOSITION 3. *Let M_1, M_2 and M_3 be positive integers. Consider the problem of minimizing*

$$(\mu,\nu) \longrightarrow \mu(f_0) + \nu(f_1)$$

over the set $S(M_1,M_2,M_3)$ of measures in $\mathcal{M}^+(\Omega)\times\mathcal{M}^+(\omega)$ satisfying $\mu(1_\Omega) \leqslant T$, $\nu(1_\omega) \leqslant T$, and (for P) $\mu(F_i) + \nu(G_i) = \alpha_i$, $i=1,2,...,M_1$, while for P′ these are replaced by: $\mu(F_i') + \nu(G_i') \leqslant \alpha_i'$, $i=1,2,...,M_1$, while for both cases: $\mu(\xi_j) = a_j$, $j=1,2,...,M_2$, $\nu(\zeta_k) = b_k$, $k=1,2,...,M_3$. Then, as $M_1,M_2,M_3 \longrightarrow \infty$,

$$\inf_{S(M_1,M_2,M_3)} [\mu(f_0) + \nu(f_1)] \longrightarrow \inf_S [\mu(f_0) + \nu(f_1)].$$

We note that our optimization is global, $\inf_S I \leqslant \inf J$, where this infimun is over the clases of admissible pairs for either P or P'; this may be an strict inequality; see [3] for a discussion of this point.

How do we construct suboptimal pairs of trajectories and controls for the functional (5)?. First we obtain optimal measures (μ^*,ν^*) for a problem such as the one in Proposition 3. We obtain then a (weak*) approximation to (μ^*,ν^*) by a set of two piecewise-constant functions (u,v) by means of the results given in Proposition 2. The control function v can serve as boundary function for a *weak solution* u_v of the system (2) for P and (4) for P'. Then, for both P and P',

THEOREM 1. *Let (u_v,v) be the pair constructed as explained above. Then, under the appropriate conditions on the approximations involved,*

i) *The pair (u_v,v) is asymptotically admissible; that is, as $M_1,M_2,M_3 \longrightarrow \infty$, the final state $u_v(\cdot,T) \longrightarrow g$ in $L_2(D_T)$.*

ii) *As M_1,M_2,M_3, tend to ∞, $J(u_v,v) \longrightarrow \inf_S I(\mu,\nu)$.*

This theorem has suggested a computational method for this type of problem; since the problems involved are semi-infinite linear programming problems, estimates for the optimal controls can be obtained by applying some of the methods of solution for such problems. The linear programs which result are well-behaved; results of these, as the proofs of the theorems, will be given in the literature ([5],[6]).

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