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Modular Bernstein Algebras [1]

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0. INTRODUCTION AND PRELIMINARIES

Let A be an algebra. We will put $L(A)$ for its lattice of subalgebras, where $\leqslant$, $\wedge$, $\vee$ are naturally defined ($\leqslant$ is the inclusion relation). We define the *length of* A (and put $l(A)$) as the length of $L(A)$. It is very easy to prove that if A is a solvable algebra, then its length and its dimension coincide. In general we have obviously that $l(A) \leqslant \dim_K A$ (if K is the ground field). Throughout this paper we will deal with algebras over infinite fields of characteristic not two.

DEFINITION. An algebra A is called *modular* (*semimodular*) if $L(A)$ is modular (semimodular).

DEFINITION. A *Bernstein K-algebra* A is a commutative finite-dimensional algebra over K for which there exists a nonzero algebra homomorphism $\omega\colon A \longrightarrow K$ such that $(x^2)^2 = (\omega(x))^2 \cdot x^2$, for any x in A.

The homomorphism ω is unique and is called the *weight homomorphism of* A. It is readily seen that, given (A,ω) a Bernstein algebra, for any e, a nonzero idempotent in A (which obviously exists), there exists a Peirce decomposition of A: $A = Ke \dotplus \operatorname{Ker}\omega$, $\operatorname{Ker}\omega = U_e \dotplus V_e$, where

$$U_e = \{x \in A \mid ex = \tfrac{1}{2}x\} = \{xe \mid x \in \operatorname{Ker}\omega\}, \quad V_e = \{x \in A \mid ex = 0\}.$$

If $r = \dim_K U_e$, $s = \dim_K V_e$, the pair $(r+1, s)$ is called the *type* of the Bernstein algebra A and does not depend on the choice of e. It is also shown that $\dim_K (U_e)^2$ and $\dim_K (U_e V_e + (V_e)^2)$ do not depend on the choice of the idempotent.

DEFINITION. A Bernstein algebra (A,ω) is called *exclusive* if $(U_e)^2 = 0$ for

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some (equivalently for all) nonzero idempotent e in A. A Bernstein algebra (A,ω) is said to be *genetic* if $\operatorname{Ker}\omega$ is nilpotent.

THEOREM. *Let A be a Bernstein algebra. Then the following are equivalent:*
(*i*) *A is modular.*
(*ii*) *A is semimodular.*
(*iii*) *For any pair of subalgebras of A, S and T, $S+T$ is a subalgebra of A.*

PROPOSITION. *Let A be a trivial Bernstein algebra. Then A is modular if and only if A has type $(n+1,0)$ or $(1,n)$.*

LEMMA. *A modular Bernstein algebra is always exclusive.*

1. MODULAR BERNSTEIN ALGEBRAS OVER ARBITRARY FIELDS

Let (A,ω) be a non-trivial Bernstein algebra. If x is an element in $\operatorname{Ker}\omega$, we can consider $\mathrm{m}(x)$, the largest positive integer such that the elements x, x^2, $\ldots, x^{\mathrm{m}(x)}$ are linearly independent. Let us fix e, a nonzero idempotent in A. If we take $0\neq v$ in V_e, it can be seen that $\mathrm{m}(v)\geqslant 2$ and moreover, from the definition of $\mathrm{m}(v)$, we get that the algebra generated by v is $Kv \dotplus Kv^2 \dotplus \ldots \dotplus Kv^{\mathrm{m}(v)}$.

PROPOSITION. *Let A be a non-trivial modular Bernstein algebra of type $(r+1,s)$ and e a nonzero idempotent in A. Then for all $0\neq v$ in V_e we have:*

$$\mathrm{m}(v)=r+1\,,\quad U_e = Kv^2 \dotplus \ldots \dotplus Kv^{r+1}.$$

THEOREM. *Let A be a non-trivial modular Bernstein algebra of type $(r+1,s)$ and e a nonzero idempotent in A. Then, for any v in V_e, $v^{r+2}=0$.*

THEOREM. *Let (A,ω) be a non-trivial Bernstein algebra of type $(r+1,s)$. Then the following are equivalent:*
(*i*) *A is a modular algebra.*
(*ii*) *A is exclusive and for all nonzero idempotent e in A and any $0\neq v$ in V_e:*

$$U_e = Kv^2 \dotplus \ldots \dotplus Kv^{r+1} \quad \text{and} \quad v^{r+2}=0\,.$$

(*iii*) *A is exclusive and there exists a nonzero idempotent e in A such that for any $0\neq v$ in V_e:*

$$U_e = Kv^2 \dotplus \ldots \dotplus Kv^{r+1} \quad \text{and} \quad v^{r+2}=0\,.$$

(*iv*) *A is exclusive and the subalgebras of A contained in* $\operatorname{Ker}\omega$ *are contained in U_e or contain U_e.*

(v) *A is exclusive and the subalgebras of A contained in* $\operatorname{Ker}\omega$ *are exactly the subspaces of* U_e *and the ideals of A contained in* $\operatorname{Ker}\omega$.

2. MODULAR BERNSTEIN ALGEBRAS OVER ALGEBRAICALLY CLOSED FIELDS

The natural question in this moment is if all modular Bernstein algebras are necessarily genetic. First of all, we see that this question has a negative answer over arbitrary fields:

EXAMPLE. Let K be $\mathbb{Q}$ and A a commutative K-algebra of dimension 6, given by:

$A = Ke \dot{+} Ku_1 \dot{+} Ku_2 \dot{+} Ku_3 \dot{+} Kv \dot{+} Kw$, where $e^2=e$, $eu_i=\frac{1}{2}u_i$, $i=1,2,3$, $ev=ew=0$, $u_iu_j=0$, $i,j=1,2,3$, $u_1v=u_2$, $u_2v=u_3$, $u_3v=0$, $u_1w=0$, $u_2w=u_1$, $u_3w=-u_2$, $v^2=u_1$, $w^2=u_3$, $vw=\frac{3}{2}(-u_1+u_3)$.

Define the linear map $\omega: A \longrightarrow K$ by $\omega(e)=1$, $\omega(u_i)=0$, $i=1,2,3$, $\omega(v)=\omega(w)=0$. It is straightforward to see that (A,ω) is a Bernstein algebra. It can be seen that (A,ω) is modular. Nevertheless, A is not genetic since $0 \neq u_1 = u_2w = (u_1v)w = ((u_2w)v)w = \dots$.

THEOREM. *Any modular Bernstein algebra over an algebraically closed field K of characteristic different from two is genetic.*

COROLLARY. *Let K be an algebraically closed field of characteristic different from two and A a non-trivial Bernstein algebra of dimension* $n+1$ *over K. Then A is modular if and only if A is isomorphic to the commutative algebra:*

$Ke \dot{+} Ku_1 \dot{+} \dots \dot{+} Ku_{n-1} \dot{+} Kv$, *where* $e^2=e$, $eu_i=\frac{1}{2}u_i$, $1 \leqslant i \leqslant n-1$, $ev=0$, $u_iu_j=0$, $1 \leqslant i,j \leqslant n-1$, $u_iv=u_{i+1}$, $1 \leqslant i \leqslant n-2$, $u_{n-1}v=0$, $v^2=u_1$.

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