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Homomorphisms on some Function Algebras

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0. INTRODUCTION

Suppose that A is an algebra of continuous real functions defined on a topological space X. We shall be concerned here with the problem as to whether every nonzero algebra homomorphism $\varphi : A \longrightarrow \mathbb{R}$ is given by evaluation at some point of X, in the sense that there exists some a in X such that $\varphi(f) = f(a)$ for every f in A. This problem goes back to the work of Michael [19], motivated by the question of automatic continuity of homomorphisms in a symmetric $*$-algebra. More recently, the problem has been considered by several authors, mainly in the case of algebras of smooth functions: algebras of differentiable functions on a Banach space in [2], [11], [13] and [14]; algebras of differentiable functions on a locally convex space in [3], [4], [5] and [6], and algebras of smooth functions in the abstract context of "smooth spaces" in [18]. We shall be interested both in the general case and in the case of functions on a Banach space.

This report is based on the results obtained in [8].

1. GENERAL RESULTS

For a topological space X, let $C(X)$ be the algebra of all continuous real functions defined on X, and let $C^*(X)$ be the subalgebra of all bounded functions in $C(X)$. If A is a subalgebra of $C(X)$, we denote by $Hom\,A$ the set of all nonzero multiplicative linear functionals on A. For each $a \in X$, let δ_a be the functional $f \mapsto \delta_a(f) = f(a)$ on A; clearly $\delta_a \in Hom\,A$. We shall write $Hom\,A = X$ when every $\varphi \in Hom\,A$ is of the form $\varphi = \delta_a$ for some $a \in X$. Recall that a subalgebra A of $C(X)$ is said to be inverse-closed (respectively, closed under bounded inversion) if whenever $f \in A$ and $f(x) \neq 0$ (respectively, $|f(x)| \geqslant 1$) for

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every $x \in X$, then $1/f \in A$.

If X is a completely regular space, let βX be the Stone–Cech compactification of X and, for $f \in C(X)$, let $\hat{f}: \beta X \longrightarrow \mathbb{R} \cup \{\infty\}$ denote the continuous extension of f. Note that if f is bounded then $\hat{f}$ is finite. For each $\xi \in \beta X$ we define the algebra $A_\xi = \{f \in C(X) : \hat{f}(\xi) \neq \infty\}$.

PROPOSITION 1.1. *Let X be a completely regular space, let $A \subset C(X)$ be a subalgebra with unit, closed under bounded inversion, and let $\varphi \in Hom A$. Then there exists $\xi \in \beta X$ such that $A \subset A_\xi$ and $\varphi(f) = \hat{f}(\xi)$ for every $f \in A$.*

REMARKS 1.2. (1) In Proposition 1.1, the point $\xi \in \beta X$ is not unique, in general. We can consider as an example the subalgebra $A \subset C(\mathbb{R})$ of all bounded uniformly continuous functions on $\mathbb{R}$. In this case each $\xi \in \beta\mathbb{R}$ defines a homomorphism on A, and, using ideas of [15], it is not difficult to find two different points in $\beta\mathbb{R}$ defining the same homomorphism on A.

(2) We cannot delete the condition "A is closed under bounded inversion" in Proposition 1.1. For instance, if $X = [0,1]$ and $A \subset C([0,1])$ is the subalgebra of all polynomial functions on $[0,1]$, then $\beta X = X$ but every $\xi \in \mathbb{R}$ defines a homomorphism on A.

(3) Let X be a completely regular space and let $A \subset C(X)$ be a subalgebra with unit. If $\varphi \in Hom A$ is positive (that is, $\varphi(f) \geqslant 0$ whenever $f \geqslant 0$) then Proposition 1.1 implies that there exists $\xi \in \beta X$ such that $A \subset A_\xi$ and $\varphi(f) = \hat{f}(\xi)$ for every $f \in A$. On the other hand, it also follows from Proposition 1.1 that, if A is closed under bounded inversion, then every $\varphi \in Hom A$ is positive.

PROPOSITION 1.3. *Let X be a completely regular space and let $A \subset C(X)$ be a subalgebra with unit, closed under bounded inversion. Suppose that for each $\xi \in \beta X \setminus X$ there exists $f \in A$ such that $\hat{f}(\xi) = \infty$. Then $Hom A = X$.*

The condition in Proposition 1.3 is quite abstract, but it can be applied directly in many cases. For example, if $A \subset C(\mathbb{R}^n)$ is a unital subalgebra closed under bounded inversion and A contains the projections $\pi_j : \mathbb{R}^n \longrightarrow \mathbb{R}$ (for $j = 1, \ldots, n$), then Proposition 1.3 implies that $Hom A = \mathbb{R}^n$. Indeed, in this case $(\pi_1^2 + \ldots + \pi_n^2)\hat{}(\xi) = \infty$ for every $\xi \in \beta\mathbb{R}^n \setminus \mathbb{R}^n$. In particular, A could be the algebra of all rational functions, or all real-analytic functions, or all C^m-functions $(1 \leqslant m \leqslant \infty)$ on $\mathbb{R}^n$. More generally, if X is locally compact,

σ-compact and noncompact, there exists $h \in C(X)$ such that $\hat{h}(\xi) = \infty$, for every $\xi \in \beta X \setminus X$; now if $A \subset C(X)$ is a unital subalgebra closed under bounded inversion and A contains a function h with this property, then $Hom A = X$.

On the other hand, Proposition 1.3 certainly applies to algebras which are not inverse-closed, as the following example shows. We recall that, with some technical modifications, an analogous example can be constructed for any realcompact non-pseudocompact space.

EXAMPLE 1.4. Let X be a locally compact, σ-compact, noncompact space. Consider $g_0 \in C(\beta X)$ such that $\beta X \setminus X = \{\xi \in \beta X : g_0(\xi) = 0\}$. Using the fact that $\beta X \setminus X$ is not a P-space (see [10]) it is possible to find $g_1 \in C(\beta X)$ and $\eta \in \beta X \setminus X$ so that $\eta \in Z = \{\xi \in \beta X \setminus X : g_1(\xi) = 0\}$ but Z is not a neighbourhood of η in $\beta X \setminus X$. Consider now the function $g = (g_0^2 + g_1^2)^{-1}|_X$, and note that $Z = \{\xi \in \beta X : \hat{g}(\xi) = \infty\}$. Now let A be the unital subalgebra of $C(X)$ generated by g and A_η, that is:

$$A = \{f_0 + f_1 g + \ldots + f_n g^n : f_0, f_1, \ldots, f_n \in A_\eta \; ; \; n \in \mathbb{N}\}.$$

The algebra A has the following properties:

(1) A is closed under bounded inversion.

(2) For each $\xi \in \beta X \setminus X$ there exists $f \in A$ such that $\hat{f}(\xi) = \infty$.

(3) $Hom A = X$.

(4) If $h \in C(X)$ satisfies $\hat{h}(\xi) = \infty$, for every $\xi \in \beta X \setminus X$, then $h \notin A$.

(5) A is not inverse-closed.

Now suppose that in Proposition 1.3 the condition on A is not fulfilled, i.e., there exists $\xi \in \beta X \setminus X$ such that $\hat{f}(\xi) \neq \infty$ for every $f \in A$. Then consider the algebra homomorphism δ_ξ on A defined by $\delta_\xi(f) = \hat{f}(\xi)$ for every $f \in A$. Suppose that, in addition, A separates points and closed sets of X (that is, if $C \subset X$ is closed and $a \in X \setminus C$, there exists $f \in A$ such that $f(a) \notin \overline{f(C)}$). Then δ_ξ is not given by evaluation at any point of X. Summarizing, we have the following.

THEOREM 1.5. *Let X be a completely regular space and let $A \subset C(X)$ be a subalgebra with unit, closed under bounded inversion, which separates points and closed sets of X. Then the following are equivalent:*

(i) $Hom A = X$.

(ii) For each $\xi \in \beta X \setminus X$ there exists $f \in A$ such that $\hat{f}(\xi) = \infty$.

Next we give a simple application for algebras of continuous functions over an arbitrary product of real lines.

COROLLARY 1.6. *Let* $X \subset \mathbb{R}^I$ *be a closed set and let* $A \subset C(X)$ *be a subalgebra with unit, closed under bounded inversion. Suppose that* $\pi_i|_X \in A$ *for each projection* $\pi_i : \mathbb{R}^I \longrightarrow \mathbb{R}$ $(i \in I)$. *Then* $Hom A = X$.

We have also the following:

PROPOSITION 1.7. *Let* X *be a realcompact space and let* $A \subset C(X)$ *be a subalgebra with unit, closed under bounded inversion. If* A *is uniformly dense in* $C(X)$, *then* $Hom A = X$.

Now Corollary 1.8 below can be obtained as an easy consequence of Proposition 1.7 and the results of Garrido-Montalvo [9] on uniform density (see also [1]). This Corollary extends Theorem 3.2 of [18] and Theorem 2 of [14]. First recall that a zero-set in X is a set of the form $Z(f) = f^{-1}(0)$, for some $f \in C(X)$. Also, for $f \in C(X)$ we denote $coz(f) = X \setminus Z(f)$.

COROLLARY 1.8. *Let* X *be a realcompact space and let* $A \subset C(X)$ *be a subalgebra with unit satisfying*:

(i) A *is closed under bounded inversion.*

(ii) If $Z_0, Z_1 \subset X$ *are (nonempty) disjoint zero-sets, then there exists* $f \in A$ *such that* $f(Z_0) = 0$ *and* $f(Z_1) = 1$.

(iii) If (f_n) *is a sequence of functions in* A *such that* $coz(f_n) \cap coz(f_m) = \emptyset$ *for* $|n-m| > 1$, *then* $\sum_{n=1}^{\infty} f_n \in A$.

Then A *is uniformly dense in* $C(X)$, *and therefore* $Hom A = X$.

Our next result follows the lines of Theorem 1 of [11].

PROPOSITION 1.9. *Let* X *be a completely regular space and let* $A \subset C(X)$ *be an inverse-closed subalgebra with unit.*

(1) *Suppose that* $(f_n) \subset A$ *is a sequence such that, for every summable sequence* (α_n) *of positive numbers,* $\sum_{n=1}^{\infty} \alpha_n f_n$ *and* $\sum_{n=1}^{\infty} \alpha_n f_n^2$ *belong to* A. *Then for each* $\varphi \in Hom A$ *there exists* $a \in X$ *such that* $\varphi(f_n) = f_n(a)$ *for all* n.

(2) *Suppose that, in addition,* (f_n) *separates the points of* X. *Then* $Hom A = X$.

2. FUNCTIONS ON BANACH SPACES

We now turn our attention to the case of functions over a real Banach space E. Let $\mathcal{P}(E)$ denote the algebra of all continuous polynomials on E and, for $j=0,1,2,\dots$, let $\mathcal{P}(^jE)$ denote the space of all continuous j-homogeneous polynomials on E. That is, each $P_j \in \mathcal{P}(^jE)$ is a function of the form $P_j(x)=T_j(x,\dots,x)$, where T_j is a continuous j-linear functional on $E\times\dots\times E$ (thus for $j=0$, P_0 is constant), and each $P\in\mathcal{P}(E)$ is a finite sum $P=P_0+P_1+\dots+P_m$, where $P_j\in\mathcal{P}(^jE)$ for $j=0,1,2,\dots,m$. Recall that a function f defined on an open subset U of E is said to be real-analytic on U if, for every $x\in U$ there exist a neighbourhood W of 0 in E and a sequence (P_j) with each $P_j\in\mathcal{P}(^jE)$, such that $f(x+h)=\sum_{j=0}^{\infty}P_j(h)$, for every $h\in W$. Now let Ω be any subset of E. We denote by $\mathcal{R}(\Omega)$ the algebra of all rational functions on Ω, that is, the functions of the form P/Q, where $P,Q\in\mathcal{P}(E)$ and $Q(x)\neq 0$ for every $x\in\Omega$. Also, we denote by $\mathcal{A}(\Omega)$ (respectively, $C^m(\Omega)$, $1\leqslant m\leqslant\infty$) the algebra of all real functions on Ω which can be extended to a real-analytic function (respectively, an m-times continuously Fréchet differentiable function) on an open subset of E containing Ω. Note that $\mathcal{R}(\Omega)\subset\mathcal{A}(\Omega)\subset C^m(\Omega)$, and they are inverse-closed subalgebras of $C(\Omega)$.

We start with special case of the separable Hilbert space $E=\ell_2$.

PROPOSITION 2.1. *Let $A\subset C(\ell_2)$ be an inverse-closed subalgebra with unit. Suppose that A contains the dual space ℓ_2^* and the polynomials $P(x)=\sum_{n=1}^{\infty}x_n^2$ and $Q(x)=\sum_{n=1}^{\infty}s_n x_n^2$, where (s_n) is a given summable sequence of positive numbers. Then $Hom A=\ell_2$.*

REMARK 2.2. Let E be a real Banach space such that there exists a sequence $(\psi_n)\subset E^*$ of norm-one functionals separating the points of E (for example, if E is separable or E is the dual of a separable space). Consider any set $\Omega\subset E$ and let $A\subset C(\Omega)$ be an inverse-closed subalgebra with unit. Suppose that A contains the dual E^* and the polynomials $P=\sum_{n=1}^{\infty}r_n^2\psi_n^2$ and $Q=\sum_{n=1}^{\infty}s_n r_n^2\psi_n^2$, where (r_n) and (s_n) are two summable sequences of positive numbers. Then it can be shown using Proposition 2.1 that $Hom A=\Omega$.

Next we give our main result. First recall that a set Γ is said to have nonmeasurable cardinal if there exists no nontrivial two-valued measure defined on the power set of Γ (see e.g. [10] or [16]).

THEOREM 2.3. *Let Ω be any subset of a real Banach space E such that there exists a continuous, linear, one-to-one operator from E into $\ell_p(\Gamma)$, for some p, $(1<p<\infty)$ and some index set Γ of nonmeasurable cardinal. Suppose that*

$A \subset C(\Omega)$ is an inverse-closed subalgebra, such that $P|_\Omega \in A$ for every $P \in \mathcal{P}(E)$. Then $Hom A = \Omega$.
In particular $Hom\mathcal{R}(\Omega) = Hom\mathcal{A}(\Omega) = Hom C^m(\Omega) = \Omega$, $(1 \leqslant m \leqslant \infty)$.

REMARKS 2.4. (1) The hypothesis on E in Theorem 2.3 is satisfied if E is a separable space, or E is the dual of a separable space, or, more generally, if E is a closed subspace of $C(K)$, where K is a compact, separable space.

(2) Recall that super-reflexive Banach spaces can be defined as those spaces admiting an equivalent uniformly convex norm (see for instance [7]). It follows from ([17], Lemma 9) that the hypothesis on E in Theorem 2.3 is also satisfied whenever E is a super-reflexive space with nonmeasurable cardinal.

(3) The requirement on the cardinality of Γ in Theorem 2.3 is very mild, since in fact it is not known whether measurable cardinals exist. On the other hand, if we suppose that Γ has measurable cardinal, it follows that $E = \ell_2(\Gamma)$ is not realcompact (see [10]). In this case let vE denote the Hewitt-Nachbin realcompactification of E. Now if $A \subset C(E)$ is a subalgebra as in Theorem 2.3, each point $\xi \in vE \setminus E$ gives a homomorphism $\varphi(f) = \hat{f}(\xi)$ on A which is not given by evaluation at any point of E.

(4) In Theorem 2.3 we cannot change the condition "A is inverse-closed" by "A is closed under bounded inversion". Consider as an example $E = \ell_2$, let Ω be the open unit ball of E and define

$$A = \{P/Q : P, Q \in \mathcal{P}(\ell_2) \text{ with } \inf_{x\in\Omega} |Q(x)| > 0\}.$$

Then $A \subset C(\Omega)$ is a subalgebra with unit, closed under bounded inversion, which contains every polynomial function on Ω. Now let $\xi \in \beta\Omega \setminus \Omega$. Then the algebra homomorphism $\varphi(f) = \hat{f}(\xi)$ on A is not given by evaluation at any point of Ω.

The result of Theorem 2.3 does not hold for arbitrary Banach spaces, as the following example shows. An analogous example can be seen in [12].

EXAMPLE 2.5. Let $E = c_0(\Gamma)$ and let $\Omega = c_0(\Gamma) \setminus \{0\}$, where Γ is uncountable. Then:
(1) For every real-analytic function $f: \Omega \longrightarrow \mathbb{R}$, there exists $\lim_{x\to 0} f(x)$.
(2) The algebra homomorphism $\varphi: \mathcal{A}(\Omega) \longrightarrow \mathbb{R}$ defined by $\varphi(f) = \lim_{x\to 0} f(x)$ is not given by evaluation at any point of Ω.

Let Ω be an open subset of $c_0(\Gamma)$, where Γ is uncountable. Since $c_0(\Gamma)$ admits C^∞-partitions of unity (see [20]), it follows from Corollary 1.8 that

$Hom\, C^m(\Omega) = \Omega$ (see also [14]). However, in the case of real-analytic functions the situation is different. In fact, combining Example 2.5 with Theorem 2.6, we can see that the shape of Ω plays a role.

THEOREM 2.6. *Let Ω be an open ball of $c_0(\Gamma)$, or let $\Omega = c_0(\Gamma)$. Suppose that $A \subset \mathcal{A}(\Omega)$ is an inverse-closed subalgebra, such that $P_{|\Omega} \in A$ for every $P \in \mathcal{P}(c_0(\Gamma))$. Then $Hom A = \Omega$.*

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