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A New Rational and Continuous Solution for the Hilbert's 17th Problem

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In this note it is presented a new rational and continuous solution for the 17^{th} Hilbert's problem who asks if an everywhere positive polynomial can be expressed as a sum of squares of rational functions. This solution (Theorem 1) improves the results in [2] in the sense that our parameterized solution is continuous and depends in a rational way on the coefficients of the problem (what it is not the case in the solution presented in [2]). Moreover our method simplifies the proof and it is easy to generalize to other situations: for example in [4 or 5] we improve the results obtained in [7] giving more general rational and continuous solutions for several instances of Positivstellensatz (Theorem 2).

The main tool used in the proofs is the Positivstellensatz for the semipolynomials introduced in [4 or 6]. If K is an ordered field and R is a real closed field containing K, a K–semipolynomial is a function obtained by a finite repetition of composition of polynomials with coeficients in K and the function absolute value. A well–known proposition assures that the set of K–semipolynomials agrees with the minimal sup–inf stable set of functions containing polynomials with coefficients in K (see for example [3]).

A detailed version of the proof presented here can be found in [4] and another proof for this result avoiding the use of the Semipolynomial Positivstellensatz will appear in [5]. We shall use freely the definitions and results presented in [4 or 6] concerning semipolynomials.

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Let $f_{n,d}(c,X)$ be the generic polynomial with degree d and n variables (c denotes the list of coefficients $c_1,\ldots,c_m$ and X the list of variables $(X_1,\ldots,X_n)$. It is a standard fact in Real Algebraic Geometry that

$$\mathbb{F}_{n,d} = \bigcup_{i=1}^{k} \bigcap_{j=1}^{n_i} \{c : R_{n,d,i,j}(c) \geqslant 0\}$$

This last equality allows us to describe the set $\mathbb{F}_{n,d}$ in the following way

$$\mathbb{F}_{n,d} = \left\{ c : \left[\sup_{i=1,\ldots,k} \{\inf\{R_{n,d,i,j}(c) : 1 \leqslant j \leqslant n_i\}\}\right] \geqslant 0 \right\}$$

So, if for every i in $\{1,\ldots,k\}$ we define

$$H_{n,d,i}(c) = \inf\{R_{n,d,i,j}(c) : 1 \leqslant j \leqslant n_i\}$$

and

$$H_{n,d}(c) = \sup\{R_{n,d,i}(c) : 1 \leqslant i \leqslant k\}$$

we have obtained the following description for the set $\mathbb{F}_{n,d}$

$$\mathbb{F}_{n,d} = \{c : H_{n,d}(c) \geqslant 0\}$$

where $H_{n,d}(c)$ is a $\mathbb{Q}$-semipolynomial. We have shown the equivalence

$$c \in \mathbb{F}_{n,d} \iff H_{n,d}(c) \geqslant 0 \iff \forall x \in \mathbb{R}^n \; f_{n,d}(c,x) \geqslant 0$$

with $f_{n,d}(c,x)$ a polynomial and $H_{n,d}(c)$ a $\mathbb{Q}$-semipolynomial.

Let us consider now $H_{n,d}(c)$ as a $\mathbb{Q}$-semipolynomial expression defined by the straight-line program that translates the definitions of $H_{n,d,1},\ldots, H_{n,d,k}$ and $H_{n,d}$. So, we can apply the Real Positivstellensatz for the $\mathbb{Q}$-semipolynomial expressions in the context $H_{n,d}$ (see [4 or 6]) to the implication

$$\forall c \in R^m \;\; \forall x \in R^n \;\; \{H_{n,d}(c) \geqslant 0 \Longrightarrow f_{n,d}(c,x) \geqslant 0\}$$

or, what is the same, to the incompatibility of the system of generalized sign conditions

$$\dot{H}_{n,d}(c) \geqslant 0 \quad , \quad f_{n,d}(c,X) < 0$$

Applying such Positivstellensatz to this system we obtain an algebraic identity that can be rewritten as

$$f_{n,d}(c,x)g(c,x) = f_{n,d}(c,x)^{2r} + h(c,x) \tag{1}$$

where h and g are $\mathbb{Q}$−semipolynomial expressions evidently positive or null under the hypothesis $H_{n,d}(c) \geqslant 0$.

Coming back again to the definitions it is easy to see that g and h are polynomials in X whose coefficients are $\mathbb{Q}$−semipolynomial expressions in c. More precisely, g and h are sum of terms

$$p_j(c)q_j(c,X)^2$$

where the $q_j(c,X)$ have the same type than g and h and the $p_j(c)$ are $\mathbb{Q}$−semipolynomial expressions evidently positive or null under $H_{n,d}(c) \geqslant 0$ and with the context $H_{n,d}(c)$. This allows us to conclude that, without loss of generality, we can suppose that every $p_j(c)$ is a product whose factors have the following type

- the $\mathbb{Q}$−semipolynomial expression $H_{n,d}(c)$,
- a $\mathbb{Q}$−semipolynomial expression

$$H_{n,d}(c) - H_{n,d,i}(c) \quad \text{or} \quad R_{n,d,i,j}(c) - H_{n,d,i}(c)$$

- a positive rational or the square of a $\mathbb{Q}$−semipolynomial expression in c.

If we multiply by $f_{n,d}(c,X)$ every member of the equality (1) we get

$$f_{n,d}(c,X) = \frac{f_{n,d}(c,X)^2 g(c,X)}{f_{n,d}(c,X)^{2r} + h(c,X)} = \frac{f_{n,d}(c,X)^2 g(c,X) k(c,X)}{k(c,X)^2} = \frac{g_1(c,X)}{k(c,X)^2}$$

denoting by $k(c,X)$ the denominator of the first fraction and where g_1 has the same type than g and h. We have proved the following theorem.

THEOREM 1. *The generic polynomial with degree d and n variables can be written as a sum of rational functions*

$$f_{n,d}(c,X) = \sum_j p_j(c) \left[\frac{q_j(c,X)}{k(c,X)}\right]^2 \qquad (*)$$

where

- *the $q_j(c,X)$ and $k(c,X)$ are polynomials in the variables X whose coefficients are $\mathbb{Q}$−semipolynomial expressions in the variables c. Moreover if $c \in \mathbb{F}_{n,d}$ then $k(c,X)$ only vanishes on the zeros of $f_{n,d}(c,X)$,*
- *each $p_j(c)$ is a product whose factors are $H_{n,d}(c)$ or one of the $\mathbb{Q}$−semipolynomial expressions $H_{n,d}(c) - H_{n,d,i}(c)$ or one of the $\mathbb{Q}$−semipolynomial expressions $R_{n,d,i,j}(c) - H_{n,d,i}(c)$ or a positive rational or the square of a*

$\mathbb{Q}$-semipolynomial expression in c. So, under the hypothesis $H_{n,d,i}(c) \geqslant 0$ the positivity of $p_j(c)$ is "clearly" evident,

- *the equality*

$$f_{n,d}(c,X)k(c,X)^2 - \Sigma\, p_j(c)q_j(c,X)^2 = 0$$

is specially evident in the following sense: the first member of the equality, as polynomial in X, has as coefficients $\mathbb{Q}$-semipolynomial expressions in c which are formally null.

The equality (*) provides a rational and continuous solution for the Hilbert's 17^{th} problem because

- *all the coefficients (the $p_j(c)$ and the coefficients of the $q_j(c,X)$ and $k(c,X)$ considered as polynomials in X) appearing in the equality (*) are rational and continuous functions in c, more precisely they are $\mathbb{Q}$-semipolynomials expressions in the variables c,*
- *every term in sum (*)*

$$p_j(c)\left[\frac{q_j(c,X)}{k\,(c,X)}\right]^2$$

is a rational function which can be extended by continuity to a semialgebraic continuous function in the semialgebraic closed set $\mathbb{F}_{n,d} \times R$.

The solution for the Hilbert's 17^{th} problem provided by theorem 1 can be seen as a particular case of Real Positivstellensatz and for this case we have just proved the existence of a solution depending on the parameters of the problem in a semipolynomial way. So what we shall do, is to generalize this result for another cases. The next theorem provides a rational and continuous solution for some cases of Real Positivstellensatz (see [4] for a detailed proof).

THEOREM 2. *Let $\mathbb{H}(c,X)$ be a system of generalized sign conditions on polynomials in $K[c,X]$, where the X_i's are considered as variables and the c_j's as parameters. If S_{H} is the semialgebraic defined by*

$$c \in S_{\mathsf{H}} \iff \forall x \in R^n \quad \mathbb{H}(c,X) \text{ is incompatible}$$

and S_{H} is locally closed then there exist $H_1(c)$ and $H_2(c)$ K-semipolynomial expressions such that

$$c \in S_{\mathsf{H}} \iff \Big[H_1(c) \geqslant 0,\ H_2(c) > 0\Big]$$

If $c \in S_{\mathbb{H}}$ *then the incompatibility of* $\mathbb{H}(c) = \mathbb{H}(c, X)$ *inside* R^n *is made obvious by a strong incompatibility*

$$\downarrow\mathbb{H}(X)\downarrow$$

with type fixed (independent of c*) and with coefficients given by* $K-$*semipolynomial expressions in* c *(which are polynomials inside the context defined by* $H_1(c)$ *and* $H_2(c)$*. Moreover*

- *the algebraic identity obtained, seen as polynomial in* X*, has a structure specially simple, more precisely, every coefficient of such identity as polynomial in* X *is a* $K-$*semipolynomial expression in* c *formally null (in particular, this* $K-$*semipolynomial expression defines the zero function of* c *without supposing* $H_1(c) \geqslant 0$ *and* $H_2(c) > 0$*),*
- *every coefficient* $p(c)$ *in the algebraic identity which must be positive or zero (resp. strictly positive) is given by a* $K-$*semipolynomial expression evidently positive or null (resp. strictly positive) under the hypothesis* $H_1(c) \geqslant 0$ *and* $H_2(c) > 0$.

In the same way that our rational and continuous solution for the Hilbert's 17^{th} problem showed in theorem 1, improves Delzell's result (see [2]), what is obtained in theorem 2 improves Scowcroft's results (see [7]) in four aspects

a–. the semialgebraic set $S_{\mathbb{H}}$ needs not to be for us, necessarily closed,

b–. the coefficients of our solution are $K-$semipolynomials in the parameters c for the hypothesis,

c–. the algebraic identity obtained, seen as polynomial in X, has a structure specially simple, its coefficients are $K-$semipolynomial expressions in c formally null,

d–. the positivity or strict positivity of the coefficients (which must satisfy such condition) in the solution is clearly evident under the hypothesis $H_1(c) \geqslant 0$ and $H_2(c) > 0$.

In [4] it is studied the constructive meaning of these results in the framework of the field $\mathbb{R}$: the field of real numbers for the Constructive Analysis (see [1]), i.e. the real numbers defined as Cauchy sequences of rational numbers. So, using the parameterized results obtained concerning the 17^{th} Hilbert's problem, it is shown how to derive the same theorem in Constructive Algebra (while the non–parameterized solution does nor allow to derive any kind of consequence).

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