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Lower Bounds for Approximations

J.L. MONTAÑA[1], L.M. PARDO[1], R. RAMANAKORAISINA[2]

[1] *Dpto. de Matem. Estad. y C., Univ. de Cantabria,* 39071 *Santander, Spain*

[2] *I.R.M.A.R., Univ. de Rennes I,* 35042 *Rennes cedex, France*

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In his seminal paper of 1968, [5], H.E. Warren used techniques from real algebraic geometry (counting the number of connected components of an algebraic set) to show lower bounds for approximations of compact classes of continuous functions. More concretely, this author studied lower bounds for the deviation of K from F, $\mathrm{D}(K,F)$, where K, F are subsets of the Banach algebra $(C(M),\|\cdot\|_\infty)$, of continuous functions on a compact topological space M. The deviation $\mathrm{D}(K,F)$, for a compact subset K of $C(M)$, measures the precision to which F approximates K and it is defined by

$$\mathrm{D}(K,F) = \sup_{g\in K}\{\inf_{f\in F}\|f-g\|_\infty\}$$

Let U be an open semialgebraic subset of $\mathbb{R}^n$. A Nash function on U is a C^∞–semialgebraic function or, equivalently, a C^ω–function, algebraic over the ring of polynomials $\mathbb{R}[X_1,\dots,X_n]$. In the sequel, $N(U)$ denotes the ring of Nash functions over U.

The complexity of an element $\varphi \in N(U)$ is defined as the minimum degree of all non zero polynomials $p(T_1,\dots,T_n,Y) \in \mathbb{R}[T_1,\dots,T_n,Y]$ such that the graph of $\varphi : U \longrightarrow \mathbb{R}$ is contained in the algebraic set:

$$\{(x_1,\dots,x_n,y) \in \mathbb{R}^{n+1} : p(x_1,\dots,x_n,y) = 0\}.$$

Assume M to be a compact space and a continuous function $\Phi : \mathbb{R}^n \times M \longrightarrow \mathbb{R}$. We define the family of continuous functions over M parametrized by Φ by fixing the coordinates in $\mathbb{R}^n$ as

$$F(\Phi) := \{\Phi(x,-) : M \longrightarrow \mathbb{R} \ : \ x \in \mathbb{R}^n\}$$

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We shall say that the family of functions $F(\Phi)$ has Nash specializations of complexity $\leqslant d$ if the functions $\Phi(-,y):\mathbb{R}^n \longrightarrow \mathbb{R}$ are Nash functions of complexity bounded by d. These families of functions strictly include those used in [5] to obtain lower bounds for Stone–Weierstrass.

In terms of algebraic complexity theory, the non–scalar cost of a semi-algebraic function $f: A \longrightarrow B$, where $A \in \mathbb{R}^p$ and $B \in \mathbb{R}^q$ are semialgebraic sets, is defined as the maximum number of non–scalar operations needed to determine if a given point $x \in \mathbb{R}^{p+q}$ belongs to the graph of f. Recall from [2] that a non–scalar operation is either a test (i.e. a comparison of a rational function g with 0 of the type $g > 0$, $g \geqslant 0$, $g = 0$) or an arithmetical operation $* \in \{\times, \div\}$ not involving constants if it is a product $\times$, without constant denominator if it is a division $\div$.

For non–negative integers $n, d \in \mathbb{N}$ and a compact subset K of $C(M)$ we shall denote by

$$\Delta_{n,d}(K) := \inf\{\mathrm{D}(K, F(\Phi))\}$$

where $F(\Phi)$ runs over all families of functions with Nash specializations on $\mathbb{R}^n$ of complexity $\leqslant d$.

Observe that $\Delta_{n,d}(K) \leqslant \mathrm{D}_{n,d}(K)$, where $\mathrm{D}_{n,d}(K)$ is the invariant introduced in [5].

Our research for lower bounds of $\Delta_{n,d}(K)$ leads to the following

DEFINITION. For a compact subset K of $C(M)$ and a non–negative integer $m \in \mathbb{N}$, the invariant $\aleph(K, m)$ will be defined as the supremum of all real numbers $\alpha \in \mathbb{R}$, $\alpha \geqslant 0$, such that there are points $y_1, \ldots, y_m \in M$ satisfying:

For every sign sequence $\epsilon = (\epsilon_1, \ldots, \epsilon_m) \in \{-1, +1\}^m$ there is f in K such that $\epsilon_i f(y_i) \geqslant \alpha$ for $i = 1, \ldots, m$.

THEOREM. *For any compact space M and for any compact subset K of $C(M)$ we have for $d \geqslant 2$*

$$\Delta_{n,d}(K) \geqslant \aleph(K, m)$$

for every $m \geqslant [8(n+1)(2n-1)\log_2 d] + 1$, where $[8(n+1)(2n-1)\log_2 d]$ is the greatest integer in the real number $8(n+1)(2n-1)\log_2 d$.

Now, for K and m as in the definition above, we measure the fact that K contains functions of arbitrary oscillation about zero on $(x_1, \ldots, x_m) \in M^m$ by means of

$$\Omega(K,x_1,\dots,x_m) = \min_{\epsilon_i=+1}\{\sup_{g\in K}\{\min_{1<i<m}\{\epsilon_i g_i(x_i)\}\}\}$$

and from Theorem 2.2, noting as in [5] that $\Omega(K,x_1,\dots,x_m) \leqslant \aleph(K,m)$, we get

COROLLARY. *Under the above notations, for* $d \geqslant 2$

$$\Delta_{n,d}(K) \geqslant \Omega(K,m) := \sup\{\Omega(K,x_1,\dots,x_m) : (x_1,\dots,x_m) \in M^m\}$$

for every $m \geqslant [8(n+1)(2n-1)\log_2 d] + 1$.

COROLLARY. *Under the above notations, if* $M \subset \mathbb{R}^k$ *is a Nash submanifold and* $\Phi : \mathbb{R}^n \times M \longrightarrow \mathbb{R}$ *is a Nash function of non–scalar cost* $\leqslant h$ *we have:*

$$\mathrm{D}(K, F(\Phi)) \geqslant \aleph(K,m)$$

for every $m \geqslant 16(n+1)(2n-1)h + 1$.

The proof of the main theorem is shown using close techniques to that used by H.E. Warren and H.S. Shapiro (see [4]), by looking for upper bounds to the cardinal of the set

$$\{\epsilon \in \{-1,+1\}^m : \mathrm{sgn}_\epsilon(\varphi) \neq \emptyset\}$$

where $\varphi = (\varphi_1,\dots,\varphi_m)$ is a Nash mapping $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $\epsilon = (\epsilon_1,\dots,\epsilon_m) \in \{-1,+1\}^m$ are the sign conditions and $\mathrm{sgn}_\epsilon(\varphi)$ is the semialgebraic set given by $\{x \in \mathbb{R}^n : \mathrm{sgn}\,\varphi_i(x) = \epsilon_i\}$. This is done noting that this number is less than or equal to the number of connected components of

$$\mathbb{R}^n - \bigcup_{i=1}^{m}\{x \in \mathbb{R}^n : \varphi_i(x) = 0\}$$

and using a version of Milnor–Thom's theorem for Nash functions shown in [3].

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