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Nonnegativity of the Cauchy Matrix and Exponential Stability of a Neutral Type System of Functional Differential Equations

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1. Introduction

In the paper proposed the question of exponential stability of the solutions of the following system is studied

$$(1)\qquad \begin{gathered} x_i'(t) + \sum_{j=1}^{n} q_{ij}(t) x_j'(g_{ij}(t)) + \sum_{j=1}^{n} p_{ij}(t) x(h_{ij}(t)) = f_i(t), \quad t \in [0,+\infty), \\ x_i(s) = x_i'(s) = 0, \quad \text{if } s < 0, \quad i = 1,\dots,n. \end{gathered}$$

To the study of the stability systems of the form (1) numerous technical problems are reduced, related with the regulation processes (see, for instance, the monograph [7]).

Introduce the following conditions (H):

H1. $f_i,\ p_{ij},\ q_{ij},\ h_{ij},\ g_{ij} : [0,+\infty) \longrightarrow \mathbb{R}^1$ are measurable essentially bounded functions.

H2. $h_{ij}(t) \leqslant t,\ g_{ij}(t) \leqslant t$ (i.e. system (1) is of retarded type).

H3. The functions $g_{ij} : [0,+\infty) \longrightarrow \mathbb{R}^1$ must satisfy the condition: $\operatorname{mes} A = 0 \implies \operatorname{mes} g_{ij}^{-1}(A) = 0$, where mes is Lebesgue's measure, A is a set in $\mathbb{R}^1$, $g_{ij}^{-1}(A) = \{t \in [0,+\infty) : g_{ij}(t) \in A\}$, $i,j = 1,\dots,n$.

H4. $\operatorname{vrai\,sup}_{t\in[0,+\infty)} \sum_{j=1}^{n} |q_{ij}(t)| < 1, \quad i = 1,\dots,n.$

2. Preliminary Notes

The stability of the solutions of equations with delay was for a long time studied by means of approaches and methods more or less generalizing well known classical methods of investigation of the stability of ordinary differential equations

(Lyapunov's functions, analysis of the location of zeros of characteristic quasipolynomials, integral inequalities, etc.). In this way significant results were obtained, which can be found, for instance, in the well known monographs [7, 9, 10].

A background for the investigations proposed is a conception suggested by Azbelev about 15 years ago. A modern version of this conception can be found in a recent paper of Azbelev and all. [1]). We shall also note that the suggested by ourselves method of investigation of stability relies on the use of differential and integral inequalities which makes this method to a certain extent similar to the method in the monograph [9].

We shall give some basic notions of the conception of [1] and then, using the terminology introduced there, we shall point out the main peculiarity of the suggested method.

Under conditions (H) the general solution of system (1) has the form

$$x(t) = \int_0^t C(t,s)f(s)\mathrm{d}s + X(t)x(0), \tag{2}$$

where $x = \mathrm{col}(x_1,\dots,x_n)$, $f = \mathrm{col}(f_1,\dots,f_n)$, $C(t,s) = [C_{ij}(t,s)]_{i,j=1}^n$ is the Cauchy matrix, $X(t) = [X_{ij}(t,s)]_{i,j=1}^n$ is the fundamental matrix of system (1) satisfying the condition $X(0) = E$ (E is the unit $n \times n$-matrix).

From representation (2) it follows that the questions of stability of system (1) are reduced to estimates of the Cauchy matrix $C(t,s)$ and the fundamental matrix $X(t)$. A particular role is played by an exponential estimate.

DEFINITION 1. The Cauchy matrix $C(t,s)$ and the fundamental matrix $X(t)$ are said to satisfy an exponential estimate if there exist positive numbers N and α such that

$$\begin{gathered} |C_{ij}(t,s)| \leqslant N\exp\{-\alpha(t-s)\},\ 0 \leqslant s \leqslant t < +\infty,\ i,j = 1,\dots,n, \\ |X_{ij}(t)| \leqslant N\exp\{-\alpha t\}. \end{gathered} \tag{3}$$

In the proposed paper the exponential estimate (3) is obtained based on the nonnegativity of the entries of the Cauchy matrix. Conditions for nonnegativity of the entries of the Cauchy matrix for a system of functional differential equations were obtained in [5]. This result in fact extends to functional differential equations the well known theorem of Wazewski formulated in other terms. The idea of the relation between the positivity of the Cauchy matrix and the exponential stability was first suggested in [3] and further elaborated in [4, 5].

We shall note that, unlike many other methods of investigation of stability, the method suggested allows to estimate

$$\overline{\lim}_{t\to+\infty}\int_0^t |C_{ij}(t,s)|\mathrm{d}s,$$

which is important in the study of the behaviour of the solutions on infinity.

3. Main Results

First consider the case when $q_{ij}(t)\equiv 0$, $t\in[0,+\infty)$, $i,j=1,\dots,n$, i.e. investigate the equation

$$\text{(4)}\qquad \begin{aligned} &x_i'(t)+\sum_{j=1}^{n} p_{ij}(t)x_j(h_{ij}(t))=f_i(t), \quad t\in[0,+\infty), \quad i,j=1,\dots,n,\\ &x_i(s)=0, \quad \text{if } s<0. \end{aligned}$$

THEOREM 1. *Let the following conditions hold:*

1. *Conditions* (H) *are met.*
2. *The nontrivial solution* x_i *of the equation*

$$\text{(5)}\qquad \begin{aligned} &x_i'(t)+p_{ii}(t)x_i(h_{ii}(t))=f_i(t), \quad t\in[0,+\infty),\\ &x_i(s)=0, \quad \text{if } s<0, \end{aligned}$$

has no zeros on $[0,+\infty)$ *for each of the scalar equations* (5) *for* $i=1,\dots,n$.

3. *There exist positive numbers* ϵ, $z_1,\dots,z_n$, *such that the following inequalities hold*

$$\text{(6)}\qquad p_{ii}(t)z_i-\sum_{\substack{j=1\\ j\neq i}}^{n}|p_{ij}(t)|z_j\geqslant\epsilon, \quad i=1,\dots,n, \quad t\in[0,+\infty).$$

Then for the Cauchy and fundamental matrices of system (4) *estimates* (3) *are valid.*

The inequality

$$\text{(7)}\qquad \int_{h_{ii}(t)}^{t} p_{ii}(s)\mathrm{d}s\leqslant\frac{1}{e}\quad, \quad t\in[0,+\infty)$$

guarantees, as shown in [6], that the nontrivial solution of equation (5) has no zeros on $[0,+\infty)$.

If in condition 3 of Theorem 1 we set $z_1=\dots=z_n=1$, then we obtain the well known result of [2]

COROLLARY 1. *If conditions* (H) *are met, and for* $i = 1,\dots,n$ *inequalities* (7) *and*

$$p_{ii}(t) \geqslant \sum_{\substack{j=1 \\ j \neq i}}^{n} |p_{ij}(t)| + \epsilon$$

hold, where $\epsilon > 0$, *then for the Cauchy and fundamental matrices of system* (4) *estimate* (3) *is valid.*

THEOREM 2. *Let the conditions of Theorem* 1 *hold, where* $\epsilon \geqslant 1$.

Then for the Cauchy matrix of system (4) *the following estimate is valid*

$$\overline{\lim}_{t\to+\infty} \int_0^t \sum_{j=1}^{n} |C_{ij}(t,s)| \mathrm{d}s \leqslant z_i \quad , \quad i = 1,\dots,n. \tag{8}$$

Denote by $(y_{1k},\dots,y_{nk})$ a solution of the system of inequalities

$$p_{ii}(t) y_{ik} - \sum_{\substack{j=1 \\ j \neq i}}^{n} |p_{ij}(t)| y_{jk} \geqslant \delta_{ik} \quad , \quad i = 1,\dots,n, \tag{9}$$

where $\delta_{ik} = \begin{cases} 1, \text{ if } i = k, \\ 0, \text{ if } i \neq k. \end{cases}$

THEOREM 3. *Let the following conditions hold:*

1. *Conditions* 1 *and* 2 *of Theorem* 1 *are met.*
2. *For any* $k \in \{1,\dots,n\}$ *there exist positive constants* $y_{1k},\dots,y_{nk}$ *satisfying inequalities* (9).

Then for the Cauchy and fumdamental matrices of system (4) *the following estimate is valid*

$$\overline{\lim}_{t\to+\infty} \int_0^t |C_{ij}(t,s)| \mathrm{d}s \leqslant y_{ij} \quad , \quad i,j = 1,\dots,n.$$

THEOREM 4. *Let the conditions of Theorem* 1 *hold, where* $\epsilon \geqslant 1$, *as well as the inequalities*

$$\operatorname{vrai\,sup}_{t\in[0,+\infty)} \sum_{j=1}^{n} |q_{ij}(t)| \left[1 + \sum_{k=1}^{n} |p_{jk}(t)| z_k\right] < 1 \quad , \quad i = 1,\dots,n.$$

Then for the Cauchy and fundamental matrices of system (1) *estimates* (3) *are valid.*

We shall note that the constants $z_1,\dots,z_n$ can be found solving the following linear algebraic system

$$p_{ii}^{*} z_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} |p_{ij}|_{*} z_j = 1 \quad , \quad i = 1,\ldots,n, \tag{10}$$

where $p_{ii}^{*} = \overline{\lim}_{t\to+\infty} p_{ii}(t)$, $|p_{ij}|_{*} = \underline{\lim}_{t\to+\infty} |p_{ij}(t)|$.

In the case when $p_{ij} = \text{const.}$, $p_{ij} \leqslant 0$, $i \neq j$, $i, j = 1,\ldots,n$, for the Cauchy matrix of system (4) estimates (3) are valid if and only if all components of the solution $z_1,\ldots,z_n$ of the linear algebraic system (10) are positive ([3]).

EXAMPLE 1. Consider the system

$$\begin{aligned} &x_1'(t) + q_{11}(t)x_1'(g_{11}(t)) + q_{12}(t)x_2'(g_{12}(t)) + 4x_1(h_{11}(t)) - x_2(h_{12}(t)) = f_1(t) \\ &x_2'(t) + q_{21}(t)x_1'(g_{21}(t)) + q_{22}(t)x_2'(g_{22}(t)) - 2x_1(h_{21}(t)) + 2x_2(h_{22}(t)) = f_2(t) \\ &\qquad t \in [0,+\infty) \quad , \quad x_i(s) = x_i'(s) = 0 \ , \text{ if } \ s < 0 . \end{aligned} \tag{11}$$

We shall assume that inequality (7) holds for $i = 1,2$.

Consider first system (11) for $q_{ij}(t) \equiv 0$, $t \in [0,+\infty)$, $i,j = 1,2$. The result of [2, Corollary 1] cannot be applied since $|p_{21}| = p_{22}$. Solving the algebraic system

$$\begin{cases} 4z_1 - z_2 = 1, \\ -2z_1 + 2z_2 = 1, \end{cases}$$

we obtain $z_1 = 1/2$, $z_2 = 1$. By virtue of Theorem 4 system (11) has a Cauchy and fundamental matrices satisfying condition (3) if

$$\operatorname{vrai\,sup}_{t\in[0,+\infty)} \sum_{j=1}^{n} |q_{ij}(t)| < \tfrac{1}{4}.$$

4. PROOF OF THE THEOREMS

Proof of the Theorem 1. Define the operators K, $\overline{K} : C^n \longrightarrow C^n$, C^n is the space of continuous vector–valued functions $x : [0,+\infty) \longrightarrow \mathbb{R}^n$, by the equalities

$$(Kx)(t) = \operatorname{col}\left[-\int_0^t K_i(t,s) \sum_{\substack{j=1 \\ j\neq i}}^{n} p_{ij}(s) x_j(h_{ij}(s)) \right]_{i=1}^{n} ,$$

$$(\overline{K}x)(t) = \operatorname{col}\left[\int_0^t K_i(t,s)\sum_{\substack{j=1\\ j\neq i}}^n |p_{ij}(s)|x_j(h_{ij}(s))\right]_{i=1}^n,$$

$$x_i(s) = 0 \ , \text{ if } \ s < 0, \quad i = 1,\dots,n,$$

$K_i(t,s)$ is the Cauchy function of the scalar equation (5).

From condition 2 by virtue of [6, Theorem 1] it follows that $K_i(t,s) > 0$ for $0 \leqslant s \leqslant t < +\infty$, $i = 1,\dots,n$. That is why the operator $\overline{K} : C^n \longrightarrow C^n$ is positive.

Henceforth for the sake of simplicity of exposition we shall assume that $h_{ij}(t) \geqslant 0$ for $t \in [0,+\infty)$ since the general case is reduced to this (see [4, 5]).

The vector $(z_1,\dots,z_n)$ is a solution of the Cauchy problem

$$x_i'(t) + p_{ii}(t)x_i(h_{ii}(t)) - \sum_{\substack{j=1\\ j\neq i}}^n |p_{ij}(t)|x_j(h_{ij}(t)) = \psi_i(t), \tag{12}$$

$$t \in [0,+\infty), \quad x_i(0) = z_i, \quad i = 1,\dots,n,$$

where $\psi_i(t) \equiv p_{ii}(t)z_i - \sum_{\substack{j=1\\ j\neq i}}^n |p_{ij}(t)|z_j$, $\quad i = 1,\dots,n, \quad t \in [0,\infty)$.

That is why this vector is also a solution of the integral equation $X - \overline{K}X = \overline{\varphi}$, where

$$\overline{\varphi}_i(t) = \int_0^t K_i(t,s)\psi_i(s)\mathrm{d}s + K_i(t,0)z_i, \quad \overline{\varphi} = \operatorname{col}(\overline{\varphi}_1,\dots,\overline{\varphi}_n).$$

By condition $\psi_i(t) \geqslant \epsilon > 0$ for $t \in [0,+\infty)$, $i = 1,\dots,n$, thus $\overline{\varphi}_i(t) > 0$ for $t \in [0,+\infty)$, $i = 1,\dots,n$.
By virtue of the theorem on the integral inequality [8] the spectral radius $\rho(\overline{K})$ of the completely continuous operator $\overline{K} : C^n \longrightarrow C^n$ is less than one. By virtue of a well known assertion [8, p. 79], $\rho(K) \leqslant \rho(\overline{K}) < 1$.

By [6] the Cauchy function of the scalar equation (5) satisfies the exponential estimate (3). That is why the function

$$\varphi_i(t) = \int_0^t K_i(t,s)f_i(s)\mathrm{d}s + K_i(t,0)x_i(0)$$

is bounded on the half–axis for each function f_i.

For any right–hand side f the solution x of system (4) is bounded on the half–axis since x in this case satisfies the equation $x = Kx + \varphi$, where $\rho(K) < 1$ ($\varphi \overset{\text{def}}{=} \operatorname{col}(\varphi_1,\dots,\varphi_n)$).

From the boundedness of the solution x for any right–hand side f by [1] there follows the assertion of Theorem 1. ∎

Proof of the Theorem 2. The vector $z=\mathrm{col}(z_1,\dots,z_n)$ is a solution of problem (12). From the formula of the representation of the solution (2) we have

$$z=\int_0^t \overline{C}(t,s)\overline{1}\mathrm{d}s+\int_0^t \overline{C}(t,s)p(s)\mathrm{d}s+\overline{C}(t,0)z,$$

where $\overline{C}(t,s)=[C_{ij}(t,s)]_{i,j=1}^n$ is the Cauchy matrix of (12), $\overline{1}=\mathrm{col}(1,\dots,1)$, $p_i=\psi_i-1$, $p=\mathrm{col}(p_1,\dots,p_n)$.

By [5, Theorem 2] $\overline{C}(t,s)\geqslant 0$ for $0\leqslant s\leqslant t<+\infty$ (here and further on the inequality is meant for all entries). Thus $z\geqslant\int_0^t \overline{C}(t,s)\overline{1}\mathrm{d}s$.

To complete the proof we shall show that $|C(t,s)|\leqslant\overline{C}(t,s)$. Let $\bar{x}=\mathrm{col}(\bar{x}_1,\dots,\bar{x}_n)$ be a solution of system (12), where $\psi_i(t)=f_i(t)$, $t\in[0,+\infty)$, $i=1,\dots,n$, and let $x_i(0)=\bar{x}_i(0)$, $i=1,\dots,n$. The solutions x and $\bar{x}$ respectively of systems (4) and (12) now satisfy the integral equations $x=Kx+\varphi$, $\bar{x}=\overline{K}\bar{x}+\varphi$. From here $x=(I-K)^{-1}\varphi$, $\bar{x}=(I-\overline{K})^{-1}\varphi$ ($I:C^n\longrightarrow C^n$ is the identity operator). Since for any $\varphi\in C^n$ we have $|K\varphi|\leqslant|\overline{K}\varphi|$, then $|x|\leqslant|\bar{x}|$ and $|C(t,s)|\leqslant\overline{C}(t,s)$. ∎

The proof of Theorem 3 is carried out in a way analogous to the proof of Theorem 2.

For the proof of Theorem 4 we shall use the W–method of Azbelev [1] and the estimates of $\int_0^t\sum_{j=1}^n|C_{ij}(t,s)|\mathrm{d}s$ obtained in Theorem 2.

Set in system (1)

$$x(t)=(Wy)(t)\overset{\mathrm{def}}{=}\int_0^t W(t,s)y(s)\mathrm{d}s,$$

where $W(t,s)=[W_{ij}(t,s)]_{i,j=1}^n$ is the Cauchy matrix of system (4). We obtain the following equation in the space of n–dimensional essentially bounded measurable functions (L_∞^n): $y+\Omega y=f$, where the operator $\Omega:L_\infty^n\longrightarrow L_\infty^n$ is given by the equality

$$(\Omega y)(t)=(Qy)(t)-(QPWy)(t)$$

and the operators $Q:L_\infty^n\longrightarrow L_\infty^n$ and $P:C^n\longrightarrow L_\infty^n$ are given by the equalities

$$(Qy)(t)=\Big[\sum_{j=1}^n q_{ij}(t)y_j(g_{ij}(t))\Big]_{i=1}^n,\qquad y(s)=0,\quad\text{if}\ \ s<0,$$

$$(Px)(t) = \left[\sum_{j=1}^{n} p_{ij}(t)x_j(h_{ij}(t))\right]_{i=1}^{n}, \quad x(s) = 0, \quad \text{if } s < 0.$$

Now using the estimates given by Theorem 2, it is easy to see that the conditions of Theorem 4 guarantee that the norm of the operator $\Omega : L_\infty^n \longrightarrow L_\infty^n$ is less than one.

Thus $y = (I + \Omega)^{-1} f \in L_\infty^n$. $x = Wy$ is bounded on the half–axis since the matrix $W(t,s)$ satisfies the exponential estimate (3) by virtue of Theorem 1. Now by [1] the Cauchy matrix $C(t,s)$ and the fundamental matrix $X(t)$ of system (1) also satisfy the exponential estimate (3).

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