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Dimension of Quasi–Metrizable Spaces and Bicompleteness Degree

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In [1] J.M. Aarts introduced the notion of strong inductive completeness degree and characterized those metrizable spaces X which have a completion Y with $\dim(Y\setminus X)\leqslant n$. Recall that a topological space X has a strong inductive completeness degree -1, $\operatorname{Icd} X=-1$, if X is Čech complete. If for any two disjoint closed subsets F and G of a space X there exists an open set U such that $F\subseteq U\subseteq X\setminus G$ and $\operatorname{Icd} B(U)\leqslant n-1$ ($B(U)$ denotes the boundary of U), then X has strong inductive completeness degree $\leqslant n$, $\operatorname{Icd} X\leqslant n$. $\operatorname{Icd} X=n$ if $\operatorname{Icd} X\leqslant n$ and $\operatorname{Icd} X\nleqslant n-1$. If $\operatorname{Icd} X\nleqslant n$ for each n, then $\operatorname{Icd} X=\infty$.

THEOREM 1. (Aarts [1]) *A metrizable space X has a completion Y with $\dim(Y\setminus X)\leqslant n$ if and only if $\operatorname{Icd} X\leqslant n$.*

This note extends Aarts' theorem to quasi–metrizable spaces. Our result is applied to some relevant nonmetrizable quasi–metrizable spaces as the Sorgenfrey line and the Michael line.

Our terminology is standard (see [2]).

A quasi–pseudometric on a set X is a nonnegative real–valued function d on $X\times X$ such that for all $x,y,z\in X$: (i) $d(x,x)=0$ and (ii) $d(x,y)\leqslant d(x,z)+d(z,y)$. If d satisfies the additional condition (iii) $d(x,y)=0 \Leftrightarrow x=y$, then d is called a quasi–metric on X. If d is a quasi–(pseudo)metric on X, then the function d^{-1}, defined on $X\times X$ by $d^{-1}(x,y)=\mathrm{d}(y,x)$, is a quasi–(pseudo)metric on X called conjugate of d. Thus, the quasi–(pseudo)metric d generates a (pseudo)metric d^* on X defined by $d^*(x,y)=\max\{d(x,y),d^{-1}(x,y)\}$. ($d$ is called separating if d^* is a metric on X).

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Each quasi–pseudometric d on X also generates a topology $T(d)$ on X which has as a base the family of d–balls $\{B_d(x,r): x \in X,\ r>0\}$ where $B_d(x,r)=\{y \in X: d(x,y)<r\}$. A topological space (X,T) is called quasi–(pseudo)metrizable if there is a quasi–(pseudo)metric d on X such that $T=T(d)$. In this case we say that d is compatible with T.

Bitopological spaces appear in a natural way when one considers the topologies $T(d)$ and $T(d^{-1})$ induced by a quasi–pseudometric d and its conjugate d^{-1}. A bitopological space is [3] an ordered triple (X,P,Q) such that X is a nonempty set and P and Q are topologies on X. (X,P,Q) is called (separated) quasi–(pseudo)metrizable if there is a (separating) quasi–(pseudo)–metric d on X such that $T(d)=P$ and $T(d^{-1})=Q$. In this case we say that d is compatible with (P,Q).

A quasi–pseudometric d on a set X is called bicomplete [2],[5], if d^* is a complete pseudometric on X. A quasi–pseudometrizable space (X,P,Q) is called bicompletely quasi–pseudometrizable if it has a compatible bicomplete quasi–pseudometric.

A separated bicompletion of a separated quasi–pseudometrizable space (X,P,Q) is a separated bicompletely quasi–pseudometrizable space (Y,P',Q') such that X is a $(P' \vee Q')$–dense subset of Y and $P'|X=P$ and $Q'|X=Q$.

It is well–known [2],[5], that every separated quasi–pseudometrizable bitopological space has a separated bicompletion.

Let (X,P,Q) be a topological space such that $(X,P\vee Q)$ is a Tychonoff space. We say that $\dim X \leqslant n$ if $\dim(X,P\vee Q) \leqslant n$. Similarly, we say that $\operatorname{Icd} X \leqslant n$ if $\operatorname{Icd}(X,P\vee Q) \leqslant n$.

THEOREM 2. *A separated quasi–pseudometrizable space (X,P,Q) has a separated bicompletion (Y,P',Q') with $\dim(Y\setminus X) \leqslant n$ if and only if $\operatorname{Icd} X \leqslant n$.*

Proof. We sketch the proof. Let (X,P,Q) be a separated quasi–pseudo–metrizable space such that $\operatorname{Icd} X=-1$. Then $(X,P\vee Q)$ is a metrizable Čech complete space and, hence, a completely metrizable space. In [4, Theorem 1] it is proved that a separated quasi–pseudometrizable space (X,P,Q) is bicompletely quasi–pseudometrizable if and only if $(X,P\vee Q)$ is completely metrizable. Hence $Y=X$. Now suppose $\operatorname{Icd} X=n \geqslant 0$. Take any quasi–pseudometric d on X compatible with (P,Q). Then, there exists a separating bicomplete quasi–pseudometric d' on a set X' such that X is a $T((d')^*)$–dense subset of X' and

$d'|X=d$ ([2],[5]). By [1, Proof of Theorem 1.2] there is a subset Y of X' such that $(Y,(d')^*)$ is a complete metric space, X is $T((d')^*)$-dense in Y and $\dim(Y\setminus X)\leqslant n$. Consequently, $(Y,T(d'),T((d')^{-1}))$ is the required separated bicompletion of (X,P,Q). The converse follows immediately from Theorem 1.

Remark. Note that if $0\leqslant \mathrm{Icd}\, X\leqslant n$, the proof given in Theorem 2 actually shows that each separated bicompletion (Y,P',Q') of (X,P,Q) satisfies $\dim(Y\setminus X)\leqslant n$.

EXAMPLE 1. Let $\mathbb{R}$ be the set of real numbers and let d be the quasi-metric defined on $\mathbb{R}$ by $d(x,y)=y-x$ if $x\leqslant y$ and $d(x,y)=1$ if $x>y$. Then $T(d)$ is the Sorgenfrey line on $\mathbb{R}$. Since d^* is the discrete metric on $\mathbb{R}$ it follows that d is a bicomplete quasi-metric on $\mathbb{R}$. Thus $\mathrm{Icd}\, X=-1$ where $X=(\mathbb{R},T(d),T(d^{-1}))$.

EXAMPLE 2. Let $\mathbb{Q}$ be the set of rational numbers. Define a quasi-metric d on $\mathbb{R}$ as follows: $d(x,y)=\min\{|x-y|,1\}$ if $x\in\mathbb{Q}$, $d(x,y)=1$ if $x\in\mathbb{R}\setminus\mathbb{Q}$ and $x\neq y$, and $d(x,x)=0$ for all $x\in\mathbb{R}$. Then $T(d)$ is the Michael line on $\mathbb{R}$. Obviously, d is not bicomplete. However $\mathrm{Icd}\mathbb{R}=0$ whenever $P=T(d)$ and $Q=T(d^{-1})$. Therefore $(\mathbb{R},P,Q)$ has a separated bicompletion (Y,P',Q') with $\dim(Y\setminus\mathbb{R})=0$.

EXAMPLE 3. Let d be the separating quasi-pseudometric on $\mathbb{R}$, defined by $d(x,y)=\max\{y-x,0\}$. Then $d^*|\mathbb{Q}$ is the usual metric on $\mathbb{Q}$ and, hence, d is not bicomplete. By Theorem 2 $(\mathbb{Q},T(d),T(d^{-1}))$ has a separated bicompletion Y such that $\dim(Y\setminus\mathbb{Q})=0$ since $\mathrm{Icd}\,\mathbb{Q}=0$.

EXAMPLE 4. Let $\mathbb{Q}$ and $\mathbb{R}$ endowed with the bitopology induced by the quasi-pseudometric d of the preceding example. Then $\mathrm{Icd}(\mathbb{Q}\times\mathbb{R}^n)=n$ (see [1, page 29]).

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