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A Survey on the Classification Problem of Factors of von Neumann Algebras

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Motivated by problems in infinite dimensional representations of groups and the mathematical foundations of quantum mechanics, Murray and von Neumann initiated the study of rings of operators in 1936. In their famous work [19], they introduced the concept of a factor and gave the classification theory of factors. To be precise, they classified them as type I (type I_n, $n \in \mathbb{N}$ or type I_∞), type II (type II_1 or type II_∞) and type III. Even though they presented a general method to construct factors of type I and II in [19], at that time they could not ascertain the existence of any type III–factors at all. Later in 1940, von Neumann modified the earlier construction and gave some examples of type III factors in [25]. Thus by 1940, examples of factors of all types had been provided and the race was on. In 1943, Murray and von Neumann gave examples of two distinct type II_1–factors, distinct in the sense that they are non–isomorphic.

By the mid fifties, Pukánsky constructed two non–isomorphic type III–factors in [28]. In the period 1963–1968, many mathematicians took up the construction of new type II_1–factors and only nine distinct type II_1–factors were known before 1969, when Dusa McDuff constructed a continuum of distinct type II_1–factors in her famous memoir [18].

Earlier, in 1967, using the type III–factors of Pukánsky [28], Powers [27] constructed a continuum of distinct type III–factors. Later in 1970, Sakai gave the construction of a continuum of type II_∞–factors.

The results of Powers motivated the work of Araki and Woods [2] on infinite tensor products of type I–factors, which, in turn, played a crucial role in motivating the study of Connes [6]. Making use of the Tomita–Takesaki theory of modular Hilbert algebras, Connes, in his famous memoir [6] which won him the

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Fields medal for that decade, classified all type III-factors in terms of type III_λ-factors for $\lambda \in [0,1]$. (In this context, we should not fail to mention that Takesaki [36] independently obtained the structure theorem for the more general type III-von Neumann algebras.)

The aim of this survey is to discuss some of these important discoveries in the classification of factors of von Neumann algebras. Our hoped-for audience is the general abstract analyst; our aim, to perphaps make the treatment of the subject in texts and monographs less intimidating.

1. Definition of a Factor

Throughout the article we restrict our attention to operators acting on a separable infinite dimensional complex Hilbert space H, unless otherwise mentioned. $L(H)$ denotes the Banach algebra of all operators on H equipped with the operator norm with composition as the multiplication. A subalgebra $\mathscr{A}$ of $L(H)$ is called a $*$-subalgebra if $T^* \in \mathscr{A}$ for each $T \in \mathscr{A}$. A $*$-subalgebra $\mathscr{A}$ of $L(H)$ closed for the norm topology is called a C^*-algebra.

The inner-product of a Hilbert space is denoted by $< \cdot , \cdot >$. Isomorphisms between Hilbert spaces are isometric surjections and hence are surjections which preserve the inner-product.

For $T \in L(H)$ and $x, y \in H$, let $p_{x,y}(T) = |<Tx, y>|$. Then the locally convex topology τ_w defined by the seminorms $\{p_{x,y} : x, y \in H\}$ is called the *weak operator topology* and it is weaker than the norm topology of $L(H)$. It is well known that these two topologies coincide if and only if H is finite dimensional.

A τ_w-closed $*$-subalgebra $\mathscr{R}$ with identity of $L(H)$ is called a *von Neumann algebra*. Note that a von Neumann algebra is necessarily a C^*-algebra with identity.

Historically, von Neumann introduced this class of operators in [23] and called it a ring of operators. But later, at Dieudonné's suggestion, they became known as von Neumann algebras (see Introduction to [9]).

For the general theory of von Neumann algebras, the classic reference is [9]. However, a thorough introduction can be found in Chapter VII of [22]; [26] proceeds at a more pedantic pace, but provides the basic facts of von Neumann algebras as well.

Given a $*$-subalgebra $\mathscr{R}$ of $L(H)$, the set $\{T \in L(H) : TR = RT,\ R \in \mathscr{R}\}$ is called the *commutant* of $\mathscr{R}$ and is denoted by $\mathscr{R}'$. The commutant $(\mathscr{R}')'$ of

$\mathscr{R}'$ is called the *double commutant* of $\mathscr{R}$ and is denoted by $\mathscr{R}''$. For a $*$-subalgebra $\mathscr{R}$ of $L(H)$, it is easy to observe that $\mathscr{R}'$ is a τ_w-closed $*$-subalgebra of $L(H)$, containing the identity operator and hence is a von Neumann algebra.

Thanks to the double commutant theorem (due to von Neumann [23]), it is possible to describe a von Neumann algebra with no topological ingredient. In fact, this is the approach adopted by Dixmier in [9].

THEOREM 1.1 (THE DOUBLE COMMUTANT THEOREM). *A $*$-subalgebra $\mathscr{R}$ of $L(H)$ is a von Neumann algebra if and only if $\mathscr{R} = \mathscr{R}''$.*

Motivated by certain problems in quantum mechanics and the theory of infinite dimensional representations of groups, Murray and von Neumann made an extensive study of operator algebras in [19]. In that context, they were led to the notion of a factor of a von Neumann algebra and to the classification of factors as type I_n, $n \in \mathbb{N}$, type I_∞, type II_1, type II_∞ and type III. In the first paper [19] in 1936, they gave a general method for constructing type I- and II-factors and thus obtained some examples of these. But, as they pointed out explicity in [19], they were not aware of the existence of any type III-factors at that time.

To give the definition of a factor, we proceed as follows. Suppose $\mathscr{C}$ is a nonvoid subset of $L(H)$. Let $R(\mathscr{C})$ be the smallest von Neumann algebra in $L(H)$, which contains $\mathscr{C}$. Since $L(H)$ itself is a von Neumann algebra and the intersection of a nonvoid family of von Neumann algebras is a von Neumann algebra, obviously $R(\mathscr{C})$ is well defined. $R(\mathscr{C})$ is called the *von Neumann algebra generated* by $\mathscr{C}$. Let Σ be the class of all von Neumann algebras on H. If we partially order Σ by inclusion, then $L(H)$ and $\mathbb{C}I$ are respectively the greatest and the smallest elements in Σ, where I is the identity operator on H. Given $\mathscr{R}_1$, $\mathscr{R}_2$ in Σ, the supremum $\mathscr{R}_1 \vee \mathscr{R}_2$ and the infimum $\mathscr{R}_1 \wedge \mathscr{R}_2$, of $\mathscr{R}_1$ and $\mathscr{R}_2$, with respect to this partial ordering, exist in Σ and are given by

$$\mathscr{R}_1 \vee \mathscr{R}_2 = R(\mathscr{R}_1, \mathscr{R}_2)$$

$$\mathscr{R}_1 \wedge \mathscr{R}_2 = \mathscr{R}_1 \cap \mathscr{R}_2 .$$

Clearly, we have

$$(\mathscr{R}_1 \vee \mathscr{R}_2)' = \mathscr{R}_1' \wedge \mathscr{R}_2' . \tag{1}$$

Now, by the double commutant theorem we also have

$$(\mathcal{R}_1 \wedge \mathcal{R}_2)' = \mathcal{R}_1' \vee \mathcal{R}_2' \ .$$

$\mathcal{R}_1$ and $\mathcal{R}_2$ are said to form a *factorisation* if $\mathcal{R}_1$ and $\mathcal{R}_2$ commute elementwise and $\mathcal{R}_1 \vee \mathcal{R}_2 = L(H)$. The notion of factors arises then as a particular case of factorisation and is given as follows:

DEFINITION 1.1. For $\mathcal{R} \in \Sigma$, suppose $\mathcal{R} \vee \mathcal{R}' = L(H)$ so that $\mathcal{R}$ and $\mathcal{R}'$ form a factorisation. Then $\mathcal{R}$ is called a *factor*.

If $\mathcal{R}$ is a factor, then by the double commutant theorem $\mathcal{R}'$ is also a factor. Moreover, as $(L(H))' = \mathbb{C}I$, by (1) and Theorem 1.1 a von Neumann algebra $\mathcal{R}$ on H is a factor if and only if its centre (namely, $\mathcal{R} \cap \mathcal{R}'$) is $\mathbb{C}I$.

Before ending this section, it is worth noting that all the definitions and results given above for $*$-algebras of operators on H also hold when H is of arbitrary dimension.

2. RELATIVE DIMENSION FUNCTION OF A FACTOR

Given a factor $\mathbf{M}$ on H, we construct a relative dimension function $D_{\mathbf{M}}$ of $\mathbf{M}$ and use the range of $D_{\mathbf{M}}$ to classify $\mathbf{M}$ as type I, II or III. We prefer to use the relative dimension functions of a factor to describe the classification instead of the normal trace, since this approach is more direct and elementary. The definitions and results mentioned in this section are found in [19, 22].

Throughout this section $\mathbf{M}$ denotes a factor on H and $P(\mathbf{M})$ is the set of all projections belonging to $\mathbf{M}$. Moreover, H can be a unitary space or a separable Hilbert space.

For two projections E and F on H, it is natural to consider E to be smaller than F in size if $\dim EH \leqslant \dim FH$, where dim denotes the dimension of the subspace. Clearly, this is equivalent to saying that there exists a linear isometry U from EH onto a closed subspace of FH. On extending U linearly to the whole of H by defining $U(H \ominus EH) = 0$, we observe that $\dim EH \leqslant \dim FH$ if and only if there exists a partial isometry $U \in L(H)$ with its initial domain EH and final domain a closed subspace of FH. This observation leads to the following:

DEFINITION 2.1. For $E, F \in P(\mathbf{M})$, we write $E \precsim F$ if there exists a partial isometry $U \in \mathbf{M}$ with its initial domain EH and final domain F_1H, where $F_1 \in P(\mathbf{M})$ and $F_1 \leqslant F$. If $F_1 = F$, then we write $E \sim F$. If $E \precsim F$ and $E \sim F$ is not true, then we write $E \precnsim F$ or simply, $E \prec F$. Obviously, "$\sim$" is an equivalence relation on $P(\mathbf{M})$.

In other words, for projections E and F in $P(\mathbf{M})$, we say $E \precsim F$ if and only if there exists a partial isometry $U \in \mathbf{M}$ such that $U^*U = E$ and $UU^* = F_1 \leqslant F$.

Note that $\dim EH = \dim FH$, if $E \sim F$. But the converse is not necessarily true since $\dim EH = \dim FH$ does not guarantee the existence of a partial isometry $U \in \mathbf{M}$ for which $U^*U = E$ and $UU^* = F$ hold.

Now we can state the following result on $\precsim$.

THEOREM 2.1. *For* $E, F \in P(\mathbf{M})$, $E \precsim F$ *and* $F \precsim E$ *imply* $E \sim F$. *Moreover, given* $E, F \in P(\mathbf{M})$, *one and only one of the relations* $E \prec F$, $E \sim F$, *or* $F \prec E$ *holds.*

Analogous to the concepts of finite and infinite sets in set theory, $E \in P(\mathbf{M})$ is said to be *finite* (relative to $\mathbf{M}$), if $E \nsim F$ for any subprojection F of E belonging to $\mathbf{M}$; i.e. if $E \sim F \leqslant E$ and $F \in P(\mathbf{M})$, then $F = E$. E is said to be *infinite* (relative to $\mathbf{M}$), if it is not finite. In this case, there exists an $F \in P(\mathbf{M})$ such that $E \sim F \lneqq E$.

The following lemma of [19] gives the basis for the definitions of a fundamental sequence and a relative dimension of $\mathbf{M}$. (See Definitions 2.2 and 2.3).

LEMMA 2.1. *Let* $E, F \in P(\mathbf{M})$, $E \neq 0$ *and* F *be finite. Then there exists a finite sequence* $\{G_i\}_1^p$ *of mutually orthogonal projections in* $\mathbf{M}$ *such that*

(i) $E \sim G_1 \sim G_2 \sim \dots \sim G_p$,

(ii) $\Sigma_1^p G_i \leqslant F$,

(iii) $F - \Sigma_1^p G_i \prec E$.

Moreover, this number p *is uniquely determined by* E *and* F, *and is denoted by* $[F/E]$.

Note that $[F/E] \in \mathbb{N} \cup \{0\}$ *and* $[F/E] = 0$ *if* $F \prec E$.

A projection $E \in \mathbf{M}$ is said to be *minimal* if for any projection $F \in \mathbf{M}$ with $F \leqslant E$ we have $F = 0$ or $F = E$. Since these projections behave differently, they are treated in a special way as in the following definition.

DEFINITION 2.2. Let $\mathfrak{S} = \{E_1, E_2, \dots\}$ be an infinite sequence in $P(\mathbf{M})$ with each $E_i \neq 0$ and finite. If $[E_i/E_{i+1}] \geqslant 2$ for all i, then $\mathfrak{S}$ is said to be a *fundamental sequence* in $\mathbf{M}$. If E is a minimal projection in $\mathbf{M}$, then also $\mathfrak{S} = \{E\}$ is called a *fundamental sequence* in $\mathbf{M}$.

We note that the minimal projections are finite in $\mathbf{M}$. In [19], Murray and von Neumann established that there exists at least one fundamental sequence in $\mathbf{M}$, whenever there exists a nonzero finite projection in $\mathbf{M}$.

Given a fundamental sequence $\mathfrak{S}$ in $\mathbf{M}$, for two finite nonzero projections E and F in $\mathbf{M}$ is defined a positive real number $\left(\frac{F}{E}\right)_{\mathfrak{S}}$ by the following:

THEOREM 2.2. *Let $\mathfrak{S} = \{E_i\}_1^{\infty}$, be a fundamental sequence in $\mathbf{M}$ and $E, F \in P(\mathbf{M})$, E, F nonzero and finite, then*

$$\lim_i \frac{[F/E_i]}{[E/E_i]} = \left(\frac{F}{E}\right)_{\mathfrak{S}}$$

exists as a positive real number, where by $\lim_i$ we mean the value at $i = 1$ when $\mathfrak{S}$ consists of a minimal projection .

The functional calculus for $(\ )_{\mathfrak{S}}$, as developed in [19], suggests the following concept.

DEFINITION 2.3. A function $D : P(\mathbf{M}) \longrightarrow [0, \infty]$ is called a *relative dimension* function of $\mathbf{M}$ if

(i) $D(0) = 0$,

(ii) $E \sim F \implies D(E) = D(F)$,

(iii) $EF = 0 \implies D(E + F) = D(E) + D(F)$,

for projections E, F in $\mathbf{M}$.

If $\mathbf{M}$ has a nonzero finite projection E, we can construct a fundamental sequence $\mathfrak{S}$ in $\mathbf{M}$ by Lemma 8.13 of [19] and define a relative dimension function $D_{\mathbf{M}}$ using $\left(\frac{F}{E}\right)_{\mathfrak{S}}$ for $F \in P(\mathbf{M})$. More precisely, we have the following:

THEOREM 2.3. *Let $\mathbf{M}$ be a factor on H. Then:*

(i) *If no nonzero finite projection belongs to $\mathbf{M}$, let*

$$D_{\mathbf{M}}(F) = \begin{cases} 0 & \text{if } F = 0 \\ \infty & \text{if } F \in P(\mathbf{M}),\ F \neq 0. \end{cases}$$

If $\mathbf{M}$ has a nonzero finite projection E, let

$$D_{\mathbf{M}}(F) = \begin{cases} 0 & \text{if } F = 0 \\ \left(\frac{F}{E}\right)_{\mathfrak{S}} & \text{if } F \in P(\mathbf{M}),\ F \neq 0 \quad ,\ F \text{ finite} \\ \infty & \text{if } F \in P(\mathbf{M}) \quad \text{and } F \text{ is infinite} \end{cases}$$

where $\mathfrak{S}$ is a fundamental sequence in $\mathbf{M}$. *In this case,* $D_{\mathbf{M}}$ *is independent of the fundamental sequence* $\mathfrak{S}$ *used in the definition.*

In both cases $D_{\mathbf{M}}$ *thus defined is a relative dimension function of* $\mathbf{M}$.

(ii) *If* D' *is another relative dimension function of* $\mathbf{M}$, *then* $D' = cD_{\mathbf{M}}$ *for some constant* $c \in (0,\infty)$.

(iii) $E \overset{\prec}{\underset{\succ}{\sim}} F \iff D_{\mathbf{M}}(E) \overset{\leqslant}{\underset{>}{=}} D_{\mathbf{M}}(F)$, *where* $E \succ F$ *if* $F \prec E$.

(iv) *The range* Δ *of* $D_{\mathbf{M}}$ *satisfies the following properties:*

(a) $\Delta \subset [0,\infty]$.
(b) $0 \in \Delta$; $\sup \Delta = t_0 > 0$ *and* $t_0 \in \Delta$.
(c) *For* $t_1, t_2 \in \Delta$, $t_2 > t_1 \Rightarrow t_2 - t_1 \in \Delta$.
(d) *If* $\{t_i\}_1^\infty \subset \Delta$ *with* $\Sigma_1^\infty t_i \leqslant t_0 = \sup \Delta$, *then* $\Sigma_1^\infty t_i \in \Delta$.

(v) *The only sets* Δ, *which satisfy* (a)$-$(d) *of* (iv), *are the following:*

(I_n) : $\Delta = \{k\delta : k = 0,1,\ldots,n\}$ *for* $n \in \mathbb{N}$, $0 < \delta < \infty$.
(I_∞) : $\Delta = \{k\delta : k = 0,1,\ldots,\infty\}$, $0 < \delta < \infty$.
(II_1) : $\Delta = \{t : 0 \leqslant t \leqslant t_0\}$, $0 < t_0 < \infty$.
(II_∞) : $\Delta = \{t : 0 \leqslant t \leqslant \infty\}$.
(III) : $\Delta = \{0,\infty\}$.

If we normalise $D_{\mathbf{M}}$ *by a suitable positive multiple* (*see* (ii)) *we can take* $\delta = 1$ *in* (I_n) *and* (I_∞) *and* $t_0 = 1$ *in* (II_1).

Then we have $\Delta = \{0,1,\ldots,n\}$ *for* (I_n); $\Delta = \{0,1,\ldots,\infty\}$ *for* (I_∞) *and* $\Delta = \{t : 0 \leqslant t \leqslant 1\}$ *for* (II_1).

By an *isomorphism* Φ from $\mathscr{A}$ onto $\mathscr{B}$, where $\mathscr{A}$ and $\mathscr{B}$ are $*$-algebras, we mean a $*$-isomorphism. If a $*$-algebra $\mathscr{A}$ satisfies a property (P) and if this property (P) holds for an isomorphic image of $\mathscr{A}$, then we say that (P) is an *isomorphism invariant*. It turns out that the range Δ of $D_{\mathbf{M}}$ is an isomorphism invariant and hence is used for the classification of factors. We have the following:

DEFINITION 2.4. A factor $\mathbf{M}$ on H is said to be of *type* I_n, $n \in \mathbb{N}$, *type* I_∞, *type* II_1, *type* II_∞ or *type* III according as the range Δ of the corresponding normalised relative dimension function $D_{\mathbf{M}}$ of $\mathbf{M}$ is given by $\Delta = \{0,1,\ldots,n\}$, $\Delta = \{0,1,2,\ldots,\infty\}$, $\Delta = \{t : 0 \leqslant t \leqslant 1\}$, $\Delta = \{t : 0 \leqslant t \leqslant \infty\}$ or $\Delta = \{0,\infty\}$, respectively. When $\mathbf{M}$ is of type I_n or I_∞, $\mathbf{M}$ is said to be of *type* I or *discrete*; when $\mathbf{M}$ is of type II_1 or II_∞, $\mathbf{M}$ is said to be of *type* II or *continuous* and finally, when **M** is of type III, **M** is said to be *purely infinite*. When **M** is not of type III, **M** is said to be *semi-finite*.

Now we can state the following:

THEOREM 2.4. *For a factor* **M** *on* H *only one of the types* I_n, I_∞, II_1, II_∞ *or* III *can occur. Moreover, any two isomorphic factors on separable Hilbert spaces are of the same type. If* **M** *is a factor, then* **M** *is of type* I (*resp. of type* II, *of type* III) *if and only if* **M′** *is of the same type.*

Note 2.1. The classification theory of Murray and von Neumann [19] has been extended later to arbitrary von Neumann algebras $\mathcal{R}$ on Hilbert spaces of arbitrary dimension and accordingly, there exist unique mutually orthogonal central projections P_1, P_2 and P_3 of $\mathcal{R}$ such that $\mathcal{R}P_1$ is of type I if $P_1 \neq 0$; $\mathcal{R}P_2$ is of type II if $P_2 \neq 0$ and $\mathcal{R}P_3$ is of type III if $P_3 \neq 0$. Besides, $P_1 + P_2 + P_3 = I$. When $P_3 = 0$, $\mathcal{R}$ is said to be semi-finite; when P_2 is infinite, $\mathcal{R}P_2$ is said to be of type II_∞. For details, the reader is referred to [9], [26], etc.

3. MATRIX REPRESENTATION OF AN OPERATOR

The results of this section play a crucial role in the sequel. Suppose $H = \Sigma_1^\infty \oplus H_i$, where all the spaces H_i are isomorphic to the fixed Hilbert space H_1. Then we can represent each $T \in L(H)$ as a matrix (T_{ij}) of operators in $L(H_1)$ as folows.

Let $U_i : H_1 \longrightarrow H_i$ be an isomorphism. Considering H_i as a closed subspace of H, it is easy to observe that the adjoint U_i^* is a linear mapping from H onto H_1 such that $U_i^*(H \ominus H_i) = 0$ and U_i^* maps H_i isometrically onto H_1. Consequently, $U_i^* U_i$ is the identity operator on H_1 and $U_i U_i^*$ is the projection P_i of H onto H_i. For $T \in L(H)$, let $T_{ij} = U_i^* T U_j$. Then $T_{ij} : H_1 \longrightarrow H_1$ is linear and $\|T_{ij}\| = \|U_i^* T U_j\| \leqslant \|T\|$. Thus $(T_{ij})_{i,j}$ is a matrix of operators in $L(H_1)$ such that $\|T_{ij}\| \leqslant \|T\|$ for all i, j.

Conversely, suppose (T_{ij}) is a matrix of operators $T_{ij} \in L(H_1)$ such that $T_{ij} = U_i^* T U_j$ for some linear mapping $T : H \longrightarrow H$. Then, for $x \in H$, we have

$$\sum_{i=1}^{\infty} \|\sum_{j=1}^{\infty} T_{ij} U_j^* x\|^2 = \sum_{i=1}^{\infty} \|\sum_{j=1}^{\infty} U_i^* T P_j x\|^2 = \sum_{i=1}^{\infty} \|U_i^* T x\|^2 =$$

$$= \sum_{i=1}^{\infty} \|U_i^* P_i T x\|^2 = \sum_{i=1}^{\infty} \|P_i T x\|^2 = \|Tx\|^2,$$

since $U_i{}^*$ is an isometry on H_i.

Thus $T \in L(H)$ if and only if there exists a constant $C > 0$ such that

$$\sum_{i=1}^{\infty} \| \sum_{j=1}^{\infty} T_{ij} U_j{}^* x \|^2 \leqslant C^2 \| x \|^2 \tag{1}$$

for $x \in H$. When (T_{ij}) with $T_{ij} = U_i{}^* T U_j$ satisfies (1), we say that (T_{ij}) is *bounded*.

Thus, the matrix (T_{ij}) of operators $T_{ij} \in L(H_1)$ with $T_{ij} = U_i{}^* T U_j$ for a linear mapping $T : H \longrightarrow H$ is bounded if and only if $T \in L(H)$. In this case, T is described as the matrix (T_{ij}) and it can be observed that

$$Tx = \sum_{j=1}^{\infty} TP_j x = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} P_i T P_j x = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} U_i U_i{}^* T U_j U_j{}^* x = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} U_i T_{ij} U_j{}^* x$$

for $x \in H$. This shows that the correspondence $T \sim (T_{ij})$ is a bijective correspondence from $L(H)$ onto all bounded matrices (T_{ij}) of operators $T_{ij} \in L(H_1)$, with $T_{ij} = U_i{}^* T U_j$.

4. Factors of Type I_n, $n \in \mathbb{N}$ and I_∞.

For a unitary space H of dimension n, $L(H)$ is a type I_n-factor. If H is separable, then $L(H)$ is a type I_∞-factor.

If $\mathbf{M}$ is a type I_n-factor on H, $n \in \mathbb{N} \cup \{\infty\}$, then there exists an orthogonal family $\{E_i\}_1^n$ of minimal equivalent projections in $\mathbf{M}$ such that $\Sigma_1^n E_i = I$. Let $H_i = E_i H$. For $T \in \mathbf{M}'$, let $T \sim (T_{ij})$, with $T_{ij} = U_{1i}{}^* T U_{1j}$, where the U_{1i} are partial isometries in $\mathbf{M}$ with the initial domain H_1 and final domain H_i. Since $T \in \mathbf{M}'$, $TE_i = E_i T$ and hence T is reduced by $E_i H$. Let $T_0 = T | E_1 H$. Then it is easy to observe that $T_{ij} = \delta_{ij} T_0$ for $i, j = 1, 2, \ldots, n$. Thus $T_0 \sim (\delta_{ij} T_0)$ (see Section 3). Since $\mathbf{M} = \mathbf{M}''$, the factor $\mathbf{M}$ on the space H must consist of all matrices $A \sim (A_{ij})$ (bounded in the sense of Section 3, if $n = \infty$) of operators $A_{ij} = U_{1i}^* A U_{1j} \in L(H_1)$, which commute with all matrices of the form $(\delta_{ij} T_0)$, where $T_0 = T | H_1$ and $T \in \mathbf{M}'$. An operational calculus of these matrices immediately shows that $T_0 A_{ij} = A_{ij} T_0$ for all i, j and hence $A_{ij} \in (\mathbf{M}' E_1)'$. But $(\mathbf{M}' E_1)'$ can be shown to coincide with $E_1 \mathbf{M} E_1$ and hence $\mathbf{M}$ consists of all the matrices of the form (A_{ij}), with $A_{ij} \in E_1 \mathbf{M} E_1$ (and bounded, if $n = \infty$). On the other hand, by the spectral theorem, the von Neumann algebra $E_1 \mathbf{M} E_1$ on H_1 is the norm closure of the linear span of all the projections in $E_1 \mathbf{M} E_1$. Since E_1 is a minimal projection, this shows that $E_1 \mathbf{M} E_1 = \mathbb{C} E_1$. Consequently, $\mathbf{M} =$

$\{(\lambda_{ij}) : \lambda_{ij}\, \mathbb{C}$ and bounded if $n = \infty\}$. Thus there exists an isomorphism $U : H \longrightarrow \Sigma_1^n \oplus H_1$ such that $UAU^{-1} = (A_{ij}) = (\lambda_{ij})$ for $A \in \mathbf{M}$. Therefore, $\mathbf{M}$ is isomorphic to $L(K)$, where $K = \{(\lambda_i)_i^n : \lambda_i \in \mathbb{C},\, \Sigma_1^n |\lambda_i|^2 < \infty\}$.

Thus we have proved the following:

THEOREM 4.1. *Suppose* $\mathbf{M}$ *is a type* I_n *factor on a unitary space* H *or on a separable Hilbert space* H *with* $n \in \mathbb{N} \cup \{\infty\}$. *Then there exists a closed subspace* H_1 *of* H *and an isomorphism* V *from* H *onto* $\Sigma_1^n \oplus H_1$ *such that* $VAV^{-1} = (A_{ij}) = (\lambda_{ij})$ *for* $A \in \mathbf{M}$. *Consequently,* $\mathbf{M}$ *is isomorphic to* $L(K)$, *with* $\dim K = n$.

From the above theorem we observe that a factor of type I_n is isomorphic to $L(K)$ with $\dim K = n$, $n \in \mathbb{N} \cup \{\infty\}$. Thus all factors of type I_n (resp., of type I_∞ on a separable Hilbert space) are isomorphic to each other.

5. STRUCTURE THEOREM FOR TYPE II_∞-FACTORS

Every type II_∞-factor can be obtained as the tensor product of a type II_1-factor and $L(H_2)$ for a suitable separable Hilbert space H_2. In fact, suppose $\mathbf{M}$ is a type II_∞-factor on H. Then, by definition, there exists $E \in P(\mathbf{M})$, $E \neq 0$ and finite, such that $E\mathbf{M}E$ is a type II_1-factor on EH. Consequently, by a well-knowm result on type II_∞-factors there is an orthogonal sequence of projections $\{E_i\}_1^\infty$ in $\mathbf{M}$ such that $I = \Sigma_1^\infty E_n$ and $E \sim E_1 \sim E_2 \sim \ldots$. Then H is isomorphic to $\Sigma_1^\infty \oplus EH$. Consequently, as discussed in Section 4, it can be shown that $\mathbf{M} = \{(A_{ij}) : A_{ij} \in E\mathbf{M}E$, and the matrix is bounded in the sense of Section 3$\}$. This matrix representation is written in the form $\mathbf{M} = E\mathbf{M}E \otimes L(H_2)$ where $H_2 = \{(\lambda_i)_1^\infty : \lambda_i \in \mathbb{C},\, \Sigma_1^\infty |\lambda_i|^2 < \infty\}$ (see §2 of Chapter I of [9]). Thus we obtain the following structure theorem of type II_∞-factors:

THEOREM 5.1. *Every type* II_∞*-factor* $\mathbf{M}$ *on a separable Hilbert space* H *is of the form* $\mathbf{M}_1 \otimes L(H_2)$ *for suitable type* II_1*-factor* $\mathbf{M}_1$, *where* H_2 *is a separable Hilbert space.*

Thus the study of type II_∞-factors is reduced to that of type II_1-factors.

6. MEASURE THEORETIC CONSTRUCTION OF TYPE I- AND TYPE II-FACTORS

In [25] von Neumann modified the construction given earlier in [19] and constructed factors of type I, II and III on a separable Hilbert space. Till the appearance of [25] the existence of a type III-factor was unknown. In this section we follow [25] and limit our attention to the construction of type I- and type II-factors only, while in the next section we shall take up the study of type III-factors.

Let $(X, \mathscr{S}, \mu)$ be a σ-finite measure space with $\mu(X) > 0$ and let $\mathscr{C}$ be an utmost countable subfamily of $\mathscr{S}$ such that $\mathscr{S}$ is the σ-algebra generated by $\mathscr{C}$, $\cup\{C : C \in \mathscr{C}\} = X$, and $\mu(C) < \infty$ for $C \in \mathscr{C}$. Further, we assume that for $x, y \in X$ such that $x \in E \Leftrightarrow y \in E$ for all $E \in \mathscr{C}$, then $x = y$. In the sequel, all the measure spaces considered are supposed to satisfy the above assumptions.

DEFINITION 6.1. Let G be any utmost countable group. We say that G is an $(X, \mathscr{S}, \mu)$-group if the following conditions hold:

(i) For each $g \in G$ there exists a bijective map $T_g : X \longrightarrow X$ given by $T_g x = xg$ such that $T_{g_2} T_{g_1} = T_{g_2 g_1}$ for $g_1, g_2 \in G$. (This implies $T_e x = x$ and $(T_g)^{-1} x = T_{g^{-1}} x$ for $x \in X$, where e is the identity of G).

(ii) For $A \subset X$ and for $g \in G$, let $Ag = \{xg : x \in A\} = T_g(A)$. Then $A \in \mathscr{S}$ implies $Ag \in \mathscr{S}$.

(iii) The measures μ_g on $\mathscr{S}$ defined by $\mu_g(A) = \mu(Ag)$ for $A \in \mathscr{S}$ and $g \in G$ are absolutely continuous with respect to μ (i.e. $\mu_g \ll \mu$ for all $g \in G$).

The following definition is essential for the construction of factors.

DEFINITION 6.2. Let G be an $(X, \mathscr{S}, \mu)$-group. We say that

(i) G is *free* if $g \neq e$ and $A = \{x \in X : xg = x\}$, then $\mu^*(A) = 0$, where μ^* is the outer measure induced by μ on $\mathscr{P}(X)$;

(ii) G is *ergodic* if $A \in \mathscr{S}$, such that $\mu(Ag \,\Delta\, A) = 0$ for all $g \in G$, implies that either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$;

(iii) G is *measurable* if there exists a σ-finite measure ν on $\mathscr{S}$ such that $\nu \equiv \mu$ (i.e. $\nu \ll \mu$ and $\mu \ll \nu$) and $\nu(A) = \nu(Ag)$ for all $A \in \mathscr{S}$ and $g \in G$ (i.e. ν is G-invariant); and

(iv) G is non-measurable if G is not measurable.

In the sequel, unless otherwise mentioned, G is assumed to be an utmost countable $(X, \mathscr{S}, \mu)$-group, which is free and ergodic.

Let

$$H_\mu^G = \{F(x,g) : X \times G \longrightarrow \mathbb{C} \text{ such that } F(\cdot, g) \text{ is } \mathscr{S}\text{-measurable for each } g \in G \text{ and } \Sigma_{g \in G} \int_X |F(x,g)|^2 d\mu(x) < \infty\},$$

with the inner product given by

$$<F_1, F_2> = \Sigma_{g\in G} \int_X F_1(x,g)\overline{F_2(x,g)}\mathrm{d}\mu(x).$$

Clearly, $H_\mu^G = \Sigma_{g\in G} \oplus L^2(\mu)$. The hypotheses on $(X, \mathscr{S}, \mu)$ imply that $L^2(\mu)$ is nontrivial and separable. As G is utmost countable, H_μ^G is either a unitary space or a separable Hilbert space.

With the aim of constructing a factor we define certain linear transformations on H_μ^G as below.

DEFINITION 6.3. Let $F(\cdot,\cdot) \in H_\mu^G$, $g_0 \in G$ and ψ be a bounded $\mathscr{S}$-measurable complex function on X. Let $\frac{\mathrm{d}\mu_g}{\mathrm{d}\mu}$ be the Radon-Nikodým derivative of μ_g with respect to μ for $g \in G$. Then we define:

(a) $(\overline{U}_{g_0}F)(x,g) = \left[\dfrac{\mathrm{d}\mu_{g_0}(x)}{\mathrm{d}\mu}\right]^{1/2} F(xg_0, gg_0)$;

(b) $(\overline{V}_{g_0}F)(x,g) = F(x, g_0^{-1}g)$;

(c) $(\overline{W}F)(x,g) = \left[\dfrac{\mathrm{d}\mu_{g^{-1}}(x)}{\mathrm{d}\mu}\right]^{1/2} F(xg^{-1}, g^{-1})$;

(d) $(\overline{L}_\psi F)(x,g) = \psi(x)F(x,g)$; and

(e) $(\overline{M}_\psi F)(x,g) = \psi(xg^{-1})F(x,g)$.

The following theorem is established in [25].

THEOREM 6.1.

(i) $\overline{U}_g$, $\overline{V}_g$, $\overline{W}$, $\overline{L}_\psi$, $\overline{M}_\psi$ *as in Definition* 6.3 *are bounded operators on* H_μ^G *and* $\overline{U}_g$, $\overline{V}_g$ *and* $\overline{W}$ *are furthermore unitary.*

(ii) *Let* $\Omega = \{\overline{U}_g, \overline{L}_\psi : g \in G,\ \psi$ as in Definition 6.3 but arbitrary$\}$ *and* $\hat{\Omega} = \{\overline{V}_g, \overline{M}_\psi : g \in G,\ \psi$ as in Definition 6.3 but arbitrary$\}$. *Then* $R(\Omega) = (\hat{\Omega})'$ *and* $R(\hat{\Omega}) = \Omega'$, *where* $\Omega' = \{T \in L(H_\mu^G) : TA = AT$ for $A \in \Omega\}$, *etc.*

(iii) $R(\Omega)$ *and* $R(\hat{\Omega})$ *are spatially isomorphic and the spatial isomorphism is implemented by* $\overline{W}$ *in the sense that the isomorphism* $\Phi : R(\Omega) \longrightarrow R(\hat{\Omega})$ *is given by* $\Phi(A) = \overline{W}A\overline{W}^{-1}$, $A \in R(\Omega)$. *Each is the communtant of the other.*

(iv) $R(\Omega)$ *and* $R(\hat{\Omega})$ *are factors (since* G *is free and ergodic).*

Notation 6.1. In the sequel we shall denote $R(\Omega)$ and $R(\hat{\Omega})$ by $\mathbf{M}(X, G, \mu)$ and $\mathbf{M}'(X, G, \mu)$, respectively.

In order to define relative dimension functions $D_{\mathbf{M}}$ and $D_{\mathbf{M}'}$ of $\mathbf{M}(X,G,\mu)$ and $\mathbf{M}'(X,G,\mu)$, we make use of the results of Section 3.

Since $H_\mu^G = \Sigma_{g\in G} \oplus L^2(\mu)$, by Section 3 every $T \in L(H_\mu^G)$ has a matrix representation of the form $(T_{g,h})_{g,h\in G}$, where each $T_{g,h}$ is a bounded linear operator on $L^2(\mu)$ and $\|T_{g,h}\| \leqslant \|T\|$ for $g,h \in G$. However, when T belongs to $\mathbf{M}(X,G,\mu)$ or $\mathbf{M}'(X,G,\mu)$ we can describe T more specifically. To this end, we define the following mappings on $L^2(\mu)$.

DEFINITION 6.4. For $f \in L^2(\mu)$ let

(A) $(U_g f)(x) = \left[\dfrac{d\mu_g(x)}{d\mu}\right]^{1/2} f(xg)$ for $g \in G$, and

(B) $(L_\psi f)(x) = \psi(x) f(x)$ for any bounded $\mathscr{S}$-measurable complex function ψ on X.

Then it is known that U_g, L_ψ are bounded operators on $L^2(\mu)$ and U_g is furthermore unitary. Recall that $L^2(\mu)$ is a unitary space or a separable Hilbert space.

Now we can describe $(T_{g,h})$ as below.

THEOREM 6.2. *Let T be a bounded operator on H_μ^G with its matrix representation $(T_{g,h})_{g,h\in G}$. Then*:

(i) *$T \in \mathbf{M}(X,G,\mu)$ if and only if $T_{g,h} = L_{\psi_{g^{-1}h}(x)} U_{h^{-1}g}$;*

(ii) *$T \in \mathbf{M}'(X,G,\mu)$ if and only if $T_{g,h} = L_{\psi_{gh^{-1}}(xh^{-1})}$ where ψ_g is a bounded $\mathscr{S}$-measurable complex function on X.*

Notation 6.2. In the terminology of Theorem 6.2, we shall write $T \simeq [[\psi_g(x)]]_{g\in G}$ for $T \in \mathbf{M}(X,G,\mu)$ (resp., for $T \in \mathbf{M}'(X,G,\mu)$). (Note that the totality of the functions $\{\psi_g : g \in G\}$ is the same for both $\mathbf{M}(X,G,\mu)$ and $\mathbf{M}'(X,G,\mu)$.)

Making use of the results in [19] and [25], we can determine the types of $\mathbf{M}(X,G,\mu)$ and $\mathbf{M}'(X,G,\mu)$, when G is measurable. In fact, the following theorem describes their type classification.

THEOREM 6.3. *Suppose G is an utmost countable, free, ergodic and measurable $(X,\mathscr{S},\mu)$-group, with $\nu \equiv \mu$, where ν is a σ-finite G-invariant measure on $\mathscr{S}$. Let ν^* be the outer measure induced by ν. Let $d\nu(x)/d\mu = k(x)$ and let $T \simeq [[\psi_g(x)]]_{g\in G}$ for $T \in \mathbf{M}(X,G,\mu)$ or for $T \in \mathbf{M}'(X,G,\mu)$. Let $D_{\mathbf{M}}(E) =$*

$\int_X \psi_e(x)k(x)\mathrm{d}\mu(x)$ *and* $D_{\mathbf{M}'}(E') = \int_X \tilde{\psi_e}(x)k(x)\mathrm{d}\mu(x)$ *where* $E \tilde{-} [[\psi_g(x)]]_{g\in G}$, $E' \tilde{-} [[\tilde{\psi_g}(x)]]_{g\in G}$, $E \in \mathbf{M}(X,G,\mu)$ *and* $E' \in \mathbf{M}'(X,G,\mu)$. *(Since* G *is ergodic, the function* $k(x)$ *is uniquely determined except for a positive constant multiple). Then the following hold:*

(i) $D_{\mathbf{M}}$ *is a relative dimension function of* $\mathbf{M}(X,G,\mu)$ *and there exists a projection* $E \in \mathbf{M}(X,G,\mu)$ *with* $0 < D(E) < \infty$. *Thus* $\mathbf{M}(X,G,\mu)$ *is a non type* III*-factor. A similar result holds for* $D_{\mathbf{M}'}$ *and* $\mathbf{M}'(X,G,\mu)$.

(ii) *If* $\nu(X) < \infty$ *and if there exists* $x \in X$ *with* $\nu^*(\{x\}) > 0$, *then there exists* $N \subset X$ *with* $\nu^*(N) = 0$ *such that the one-point sets* $\{y\} \in \tilde{\mathcal{S}}$ *and* $\nu^*(\{y\}) = \nu^*(\{x\})$ *for all* $y \in X \setminus N$, *where* $\tilde{\mathcal{S}}$ *is the Lebesgue completion of* $\mathcal{S}$ *with respect to* ν. *Thus we can take* $X = \{x_1, x_2, \ldots, x_n\}$ *(say) with* $\nu^*(\{x_i\}) = \nu^*(\{x_j\}) = \epsilon$ *for* $i \neq j$, *with* $0 < \epsilon < \infty$. *Then* $\mathbf{M}(X,G,\mu)$ *and* $\mathbf{M}'(X,G,\mu)$ *are of type* I_n. *Besides,* $(1/\epsilon)D_{\mathbf{M}}$ *and* $(1/\epsilon)D_{\mathbf{M}'}$ *are the normalised relative dimension functions of* $\mathbf{M}(X,G,\mu)$ *and* $\mathbf{M}'(X,G,\mu)$, *respectively.*

(iii) *If* $\nu(X) = \infty$ *and* $\nu^*(\{x\}) > 0$ *for some* $x \in X$, *then a result similar to* (ii) *holds with* $X = \{x_i\}_1^\infty$ *and* $\nu^*(\{x_i\}) = \nu^*(\{x_j\})$ *for* $i \neq j$. *(Note that* $\nu^*(\{x\}) < \infty$*). Consequently,* $\mathbf{M}(X,G,\mu)$ *and* $\mathbf{M}'(X,G,\mu)$ *are of type* I_∞.

(iv) $\nu^*(\{x\}) = 0$ *for each* $x \in X$, *then* $\mathbf{M}(X,G,\mu)$ *and* $\mathbf{M}'(X,G,\mu)$ *are of type* II_1 *if* $\nu(X) < \infty$ *and of type* II_∞ *if* $\nu(X) = \infty$. *When* $\nu(X) < \infty$, $(1/\nu(X))D_{\mathbf{M}}$ *and* $(1/\nu(X))D_{\mathbf{M}'}$ *are the normalised relative dimension functions of* $\mathbf{M}(X,G,\mu)$ *and* $\mathbf{M}'(X,G,\mu)$, *respectively.*

With the general construction established in the above, following [19] we now give some examples of type II-factors

EXAMPLE 6.1 (Type II-factors). Let X be the set $X_\infty = \mathbb{R}$ or the set $X_1 = [0,1)$, the set $\mathbb{R}$ mod 1. Let $\mathcal{S} = \mathcal{B}(X)$, the σ-algebra of all Borel sets in X and let μ be the Borel restriction of the Lebesgue measure.

We take G to be one of the following additive groups.

(α) $G_\Theta = \{m + n\Theta : m,n \in \mathbb{Z}\}$, Θ an irrational number.

(β) $G_{\mathrm{rat}} = \{$all rational numbers in $\mathbb{R}\}$.

(γ) $G_{\mathrm{rat},p} = \{m/p^n : m \in \mathbb{Z}, n = 0,1,2,\ldots\}$, where p is any given number $2,3,\ldots$ (not necessarily prime!).

For $g \in G$, let $xg = x + g$ for $x \in X_\infty$ and $xg = x + g$ (mod 1) for $x \in X_1$.

Then it can be shown that G is a free, ergodic $(X,\mathscr{S},\mu)$-group. Since μ is translation invariant in X_∞ as in X_1, it follows that G is measurable with $\nu=\mu$. Thus by Theorem 6.3, $\mathbf{M}(X_\infty,G,\mu)$ and $\mathbf{M}'(X_\infty,G,\mu)$ are type II_∞-factors, while $\mathbf{M}(X_1,G,\mu)$ and $\mathbf{M}'(X_1,G,\mu)$ are type II_1-factors.

Note 6.1. Since the family of the groups G_θ is uncountable, apparently we have given above a continuum of type II_1 and type II_∞-factors on a separable Hilbert space. But all the type II_1-factors given in Examples 6.1 turn out to be spatially isomorphic to each other. (See Section 8).

7. Construction of Type III-Factors

When the $(X,\mathscr{S},\mu)$-group G is non-measurable, von Neumann showed in [25] that the factors $\mathbf{M}(X,G,\mu)$ and $\mathbf{M}'(X,G,\mu)$ of Notation 6.1 are of type III. The following result gives a sufficient condition for G to be non-measurable.

THEOREM 7.1. *Suppose that G is a countable $(X,\mathscr{S},\mu)$-group which is free and ergodic. Let $G_0=\{g\in G:\mu(A)=\mu(Ag)$ for all $A\in\mathscr{S}\}$. Then G_0 is a free $(X,\mathscr{S},\mu)$-group and is measurable with $\nu=\mu$. If G_0 is ergodic and $G_0\neq G$, then G is non-measurable.*

THEOREM 7.2. *If G is a free, ergodic, non-measurable $(X,\mathscr{S},\mu)$-group, then the factors $\mathbf{M}(X,G,\mu)$ and $\mathbf{M}'(X,G,\mu)$ of Section 6 (see Notation 6.1) are of type* III.

Making use of Theorems 7.1 and 7.2, the following example of a type III-factor on a separable Hilbert space is given in [25].

EXAMPLE 7.1 (A type III-factor). Let $X=\mathbb{R}$ and $\mathscr{S}=\mathscr{B}(\mathbb{R})$, the σ-algebra of all Borel subsets of $\mathbb{R}$. Let μ be the Borel restriction of the Lebesgue measure in $\mathbb{R}$. We take G to be the group of transformations $\{T(\rho,\sigma):\rho>0,$ ρ,σ rational$\}$, where

$$T(\rho,\sigma)x=\rho x+\sigma,\quad x\in\mathbb{R}$$

and the group operation of G is given by composition of transformations. Clearly, G is a free $(X,\mathscr{S},\mu)$-group and is countably infinite. The group G_0 of Theorem 7.1 is given by

$$G_0=\{T(\rho,\sigma):\mu(T(\rho,\sigma)A)=\mu(A)\quad\text{for } A\in\mathscr{S}\}=$$
$$=\{T(\rho,\sigma):\rho\mu(A)=\mu(A)\quad\text{for } A\in\mathscr{S}\}=$$

$$= \{T(1,\sigma) : \sigma \text{ rational}\}$$

and hence $G_0 \neq G$. Besides, G_0 is isomorphic to G_{rat} (see Examples 6.1) which is ergodic and hence G_0 is also ergodic. Therefore, by Theorem 7.1, G is non-measurable and hence, by Theorem 7.2, $\mathbf{M}(X, G, \mu)$ and $\mathbf{M}'(X, G, \mu)$ are type III -factors.

Note that $\mathbf{M}(X, G, \mu)$ and $\mathbf{M}'(X, G, \mu)$ are spatially isomorphic by Theorem 6.1 (iii).

Before proceeding further, we make some comments on [19] and [25]. In [19], Murray and von Neumann gave the type classification theory of factors on H and constructed the factors $\mathbf{M}(X, G, \mu)$ and $\mathbf{M}'(X, G, \mu)$ assuming that G is an utmost countable $(X, \mathscr{S}, \mu)$-group, which is free and ergodic such that $\mu(Ag) = \mu(A)$ for all $A \in \mathscr{S}$. In other words, in the terminology of Theorem 6.3, they assumed $\nu = \mu$ and hence were led to the construction of factors of type I and II only. At that time, they wondered whether any type III-factor existed at all. It was only in 1940 that von Neumann modified the construction given in [19] introducing the terminology of measurable and non-measurable $(X, \mathscr{S}, \mu)$-group and thus obtained in [25] the construction of factors of type I, II, III. These results have been described above in Section 6 and in the present section.

8. Hyperfinite Type II_1-Factors

In [21] Murray and von Neumann answered affirmatively the question whether there were at least two non-isomorphic type II_1-factors on H. This was done by studying the class of type II_1-factors known as approximately finite type II_1-factors. The main results of [21] will be presented in this section as well as in the next two sections. Here we restrict our study to isomorphism property of these factors and show that the type II_1-factors in Examples 6.1 are spatially isomorphic.

Definition 8.1. A factor $\mathscr{R}$ on H is said to be hyperfinite (=approxi-ately finite or ATI = almost type I) if there exists an increasing sequence $(M_i)_1^\infty$ of discrete factors M_i of finite type I_{n_i} (so that n_i divides n_{i+1}) such that $\mathscr{R}$ is the von Neumann algebra generated by $\bigcup_1^\infty M_i$.

Murray and von Neumann use the terminology "approximately finite" and Dixmier [9] calls it hyperfinite, which is also referred to as ATI by Connes.

When $\mathscr{R}$ is a type II_1-factor, the sequence (n_i) involved in Definition 8.1

has no role in determining its algebraic type. In fact, the following result was obtained in [21].

THEOREM 8.1. *Hyperfinite type* II_1*-factors exist on* H *and any two hyperfinite type* II_1*-factors on separable Hilbert spaces are isomorphic.*

In Theorem 6.3 we can guarantee that $\mathbf{M}(X,G,\mu)$ and $\mathbf{M}'(X,G,\mu)$ are hyperfinite when G satisfies some more conditions. In fact, the following theorem has been given in [21].

THEOREM 8.2. *Suppose in Theorem* 6.3 *the* $(X,\mathscr{S},\mu)$*-group* G *satisfies one of the following conditions*:

(*) *There exists a sequence* $G_1 \subset G_2 \subset \dots$ *of finite subgroups of* G *such that* $G = \bigcup_1^\infty G_i$.

(**) G *is abelian.*

Then the factors $\mathbf{M}(X,G,\mu)$ and $\mathbf{M}'(X,G,\mu)$ *are hyperfinite type* II_1*-factors, whenever they are of type* II_1.

A detailed proof of Theorem 8.2 corresponding to the condition (*) is found in [21], but the proof corresponding to (**) was postponed for a future publication, but was never published. Nevertheless, later in 1963 Dye [11] obtained the same result as a particular case of a more general situation.

Returning to the factors $\mathbf{M}(X_1,G_\Theta,\mu)$, $\mathbf{M}(X_1,G_{\mathrm{rat}},\mu)$ and $\mathbf{M}(X_1,G_{\mathrm{rat},p},\mu)$ of Examples 6.1, we observe that they are hyperfinite type II_1-factors by Theorem 8.2 as the groups are abelian (while G_{rat} and $G_{\mathrm{rat},p}$ also satisfy (*)) and hence by Theorem 8.1 they are isomorphic. Now by Theorem XI of [19], Theorem XI of [20] and by the isomorphism between these factors we deduce the following:

COROLLARY 8.1. *The factors* $\mathbf{M}(X_1,G,\mu)$ *of Examples* 6.1 *are spatially isomorphic hyperfinite type* II_1*-factors, where* G *is any one of the groups* G_Θ, G_{rat} *and* $G_{\mathrm{rat},p}$.

9. A SIMPLE GROUP-THEORETIC CONSTRUCTION OF TYPE II_1-FACTORS

Imposing a stringent condition on the group G, Murray and von Neumann gave a simplified version of the measure theoretic construction of Section 6 in [21] to obtain type II_1-factors. Before explaining this construction, it is worth pointing out that this construction played a crucial role in the later works of Dusa McDuff [17, 18] and Sakai [30, 31] to obtain a continuum of non-isomorphic type

II_1 and type III-factors. See Sections 12 and 13.

Suppose $X=\{x_0\}$, $\mathscr{S}=\{\{x_0\},\emptyset\}$ and $\mu(\emptyset)=0$, $\mu(\{x_0\})=1$. Given a countably infinite group G, let $x_0g=x_0$ for all $g\in G$, so that G is an $(X,\mathscr{S},\mu)$-group. In this case H_μ^G reduces to the separable Hilbert space $\ell^2(G)$, which is given by

$$\ell^2(G)=\{f\colon G\longrightarrow \mathbb{C} \text{ such that } \sum_{g\in G}|f(g)|^2<\infty\}$$

with the inner product

$$<f_1,f_2>=\sum_{g\in G}f_1(g)\overline{f_2(g)}.$$

Moreover, in this case the unitary operators $\overline{U}_{g_0}$, $\overline{V}_{g_0}$, $\overline{W}$ of Definition 6.3 assume the simple forms $\hat{U}_{g_0}$, $\hat{V}_{g_0}$, $\hat{W}$, respectively, where

$$(\hat{U}_{g_0}f)(g)=f(gg_0)$$
$$(\hat{V}_{g_0}f)(g)=f(g_0^{-1}g)$$

and

$$(\hat{W}f)(g)=f(g^{-1})$$

for g_0, $g\in G$ and $f\in\ell^2(G)$.

Then by Theorem 6.1 (i), $\hat{U}_{g_0}$, $\hat{V}_{g_0}$ and $\hat{W}$ are unitary operators on $\ell^2(G)$. As the bounded $\mathscr{S}$-measurable functions on X now reduce to constant functions the von Neumann algebras $R(\Omega)$ and $R(\hat{\Omega})$ of Theorem 6.1 (ii) are the same as those generated by $\{\hat{U}_g : g\in G\}$ and $\{\hat{V}_g : g\in G\}$, respectively. Let us denote them by $\mathfrak{U}(G)$ and $\mathfrak{V}(G)$, respectively. Then $\mathfrak{U}(G)$ and $\mathfrak{V}(G)$ are spatially isomorphic to each other by $\hat{W}$ and one is the commutant of the other. Since these algebras play an important role in the construction of type II_1- and type III-factors of later sections, we give the following:

Notation 9.1. $\mathfrak{U}(G)$ and $\mathfrak{V}(G)$ denote the von Neumann algebras generated by $\{\hat{U}_g : g\in G\}$ and $\{\hat{V}_g : g\in G\}$, respectively.

THEOREM 9.1. $(\mathfrak{U}(G))'=\mathfrak{V}(G)$ *and* $(\mathfrak{V}(G))'=\mathfrak{U}(G)$. *Moreover,* $\mathfrak{U}(G)$ *and* $\mathfrak{V}(G)$ *are spatially isomorphic and the spatial isomorphism is implemented by* $\hat{W}$.

Since $X=\{x_0\}$, with $\mu(\{x_0\})=1$ and $x_0g=x_0$ for all $g\in G$, evidently G is neither free nor ergodic. Thus it is necessary to find some other conditions on

G to ensure that $\mathfrak{U}(G)$ and $\mathfrak{V}(G)$ are factors. To this end, Murray and von Neumann introduced the following concept in [21]:

DEFINITION 9.1. A group G is called an infinite conjugacy class group (in abbreviation, an ICC-group) if the conjugate class $C_g = \{h^{-1}gh : h \in G\}$ is infinite for each $g \neq e$ in G.

Obviously, an ICC-group is a non-commutative infinite group.

Now we can state the following interesting theorem:

THEOREM 9.2. *For a countable group G, $\mathfrak{U}(G)$ and $\mathfrak{V}(G)$ are factors on the separable Hilbert space $\ell^2(G)$ if and only if G is an* ICC*-group. In this case, $\mathfrak{U}(G)$ and $\mathfrak{V}(G)$ are type* II_1*-factors. Moreover, if G satisfies condition* (*) *of Theorem* 8.2, *then these are hyperfinite type* II_1*-factors.*

As an application of the last part of the above theorem, is given below an example of a hyperfinite type II_1-factor as a $\mathfrak{U}(G)$.

EXAMPLE 9.1. Suppose that G is the subgroup of the permutation group of $\mathbb{N}$ formed by all those permutations which leave all but a finite number of elements fixed. Then G is an ICC-group and $G = \bigcup_1^\infty G_n$ with $G_n \uparrow$, where G_n is the subgroup of all those permutations which leave all but $\{1,2,\ldots,n\}$ fixed. Consequently, by Theorem 9.2 the factors $\mathfrak{U}(G)$ and $\mathfrak{V}(G)$ are hyperfinite type II_1-factors.

10. EXAMPLE OF A NON-HYPERFINITE TYPE II_1-FACTOR

All the type II_1-factors constructed in the earlier sections turn out to be hyperfinite and thus are isomorphic to each other by Theorem 8.1. Then the following question arises naturally. Does there exist any non-hyperfinite type II_1-factor on H?. Murray and von Neumann answered this question affirmatively in [21] by introducing an isomorphism invariant called "the property Γ" and then constructing a factor on H without the property Γ.

DEFINITION 10.1. We say that a type II_1-factor $\mathbf{M}$ on H has the property Γ if for each $\epsilon > 0$ and for each finite set $\{T_1, T_2, \ldots, T_n\}$ of elements in $\mathbf{M}$ there exists a unitary $U = U(T_1, T_2, \ldots, T_n) \in \mathbf{M}$ with $\mathrm{Tr}_{\mathbf{M}}(U) = 0$ and

$$[[U^{-1}T_kU - T_k]] < \epsilon \quad \text{for } k = 1,2,\ldots,n$$

where $[[A]] = (\mathrm{Tr}_{\mathbf{M}}(A^*A))^{1/2}$ and $\mathrm{Tr}_{\mathbf{M}}$ is the relative trace of $\mathbf{M}$.

Here the relative trace $\mathrm{Tr}_{\mathbf{M}}$ is an extension of $D_{\mathbf{M}}$ to all hermitian elements in $\mathbf{M}$ with $\mathrm{Tr}_{\mathbf{M}}(I) = 1$ and satisfying certain properties. (See [21]).

THEOREM 10.1. *The property* Γ *is an isomorphism invariant. If* $\mathbf{M}$ *is a hyperfinite type* II_1*-factor on* H*, then* $\mathbf{M}$ *satisfies the property* Γ. *Thus all hyperfinite type* II_1*-factors on separable Hilbert spaces satisfy the property* Γ.

In [21] Murray and von Neumann introduced a sufficient condition on the ICC-group G to ensure that $\mathfrak{U}(G)$ is not hyperfinite. Let us state their result.

THEOREM 10.2. *Let* G *be a countable* ICC*-group and suppose there exists a set* $F \subset G$ *with the following properties*:

(i) *There exists a* $g_1 \in G$ *such that* $F \cup g_1 F g_1^{-1} = G \setminus \{e\}$.

(ii) *There exists a* $g_2 \in G$ *such that the sets* F, $g_2 F g_2^{-1}$ *and* $g_2^{-1} F g_2$ *are disjoint.*

Then the factors $\mathfrak{U}(G)$ *and* $\mathfrak{V}(G)$ *do not possess the property* Γ.

Applying Theorems 10.1 and 10.2, Murray and von Neumann gave the following example of a non-hyperfinite type II_1-factor in [21].

EXAMPLE 10.1 (A non-hyperfinite type II_1-factor). Let G be the free group generated by two elements a and b. Clearly, G is a countable ICC-group. Let F be the set of those $g \in G$ which when written as a power product of a and b of minimum length end with a^n, $n = \pm 1, \pm 2, \ldots$. It is an easy exercise to verify the properties (i) and (ii) of Theorem 10.2 for the set F. Consequently, by Theorem 10.2 the type II_1-factors $\mathfrak{U}(G)$ and $\mathfrak{V}(G)$ do not satisfy the property Γ and hence are non-hyperfinite by Theorem 10.1.

The above example, and Theorems 8.1 and 10.1 imply the following

THEOREM 10.3. *There exist at least two non-isomorphic type* II_1*-factors on a separable Hilbert space* H, *one being hyperfinite and the other non-hyperfinite.*

Though Murray and von Neumann could provide more examples of non- hyperfinite type II_1-factors in [21], they could only establish the existence of just two non-isomorphic type II_1-factors in terms of the property Γ. However, their method and ideas were used later by Dixmier and Lance [10] and Dusa McDuff [17, 18], the latter even being successful in constructing an uncountable family of non-isomorphic type II_1-factors on a separable Hilbert space. See Section 12.

11. PUKÁNSKY'S EXAMPLES OF TWO NON-ISOMORPHIC TYPE III-FACTORS

Though von Neumann constructed some type III-factors on a separable Hilbert space H in [25], he did not study any isomorphism invariant to obtain non-isomorphic type III-factors. The first contribution in this direction came from Pukánsky [28], who introduced an isomorphism invariant called "the property (L)" and constructed two non-isomorphic type III-factors, one satisfying (L) and the other failing (L). Later, these examples played a fundamental role in the construction of an uncontable family of non-isomorphic type III-factors given by Powers [27], Sakai [30] and Connes [6]. See Sections 13, 14 and 16.

Following Pukánsky [28] and [32], we present the construction of these factors of Pukánsky. For details, the reader is referred to [32].

DEFINITION 11.1. A von Neumann algebra $\mathscr{R}$ on H is said to satisfy the property (L) if there exists a sequence of unitary elements $(U_k)_k^{\infty}$ in $\mathscr{R}$ such that $U_k \longrightarrow 0$ in τ_w and $\|U_k A U_k^* - A\| \longrightarrow 0$ as $k \longrightarrow \infty$ for every $A \in \mathscr{R}$.

EXAMPLE 11.1 (Type III-factor $\mathscr{M}_{(p_n)_1^{\infty}}$). Let $\Omega_n = \{0,1\}$, $n \in \mathbb{N}$ and let $X = \Pi_1^{\infty} \Omega_n$. Let μ_n be the measure on $\mathscr{P}(\Omega_n)$ defined by $\mu_n(\{0\}) = (1-p_n)/2$ and $\mu_n(\{0\}) = (1+p_n)/2$, where $0 < \delta < p_n < 1-\delta$ for some fixed $\delta > 0$. Let $\mu = \Pi_1^{\infty}\mu_n$ be the product measure on the corresponding σ-algebra in X. Let $G = \{w = (w_n)_1^{\infty} : w_n \neq 0$ occurs for a finite number of n's only$\}$. Then, with respect to addition given coordinatewise mod 2, G is a countable group. For $g \in G$ and $w \in X$, let $wg = w + g$, where $(w+g)_i = w_i + g_i$ (mod 2). Then it can be shown that G is a free, ergodic, non-measurable $(X, \mathscr{S}, \mu)$-group. Consequently, by Theorem 7.2, $\mathbf{M}(X,G,\mu)$ and $\mathbf{M}'(X,G,\mu)$ are type III-factors on the separable space H_{μ}^{G}. Moreover, these factors satisfy the property (L) if $(1-p_n)/2 = p$ and $(1+p_n)/2 = q$ for all n. (See [28]). For later use, let us denote $\mathbf{M}(X,G,\mu)$ corresponding to $(p_n)_1^{\infty}$ by $\mathscr{M}_{(p_n)_1^{\infty}}$.

EXAMPLE 11.2 (Type III-factor $\mathbb{P}$). Let G be the free group generated by two elements. Then G is countably infinite. For each $g \in G$, let $X_g = \{0,1\}$. Let μ_g be the measure defined by $\mu_g(\{0\}) = p$ and $\mu_g(\{1\}) = q$ with $0 < p < q$ and $p + q = 1$. Let $X = \Pi_{g \in G} X_g$ and let $(X, \mathscr{S}, \mu)$ be the associated product measure space. Let

$$G_0 = \{x = (x_g)_{g \in G} : x_g \neq 0 \text{ for a finite number of } g\text{'s only}\}.$$

Let $\mathscr{G} = \{(x,g) : x \in G_0,\ g \in G\}$. For each element $\alpha = (x^0,g_0) \in \mathscr{G}$, let us define the transformation $T_\alpha : X \longrightarrow X$ given by

$$T_\alpha x = x\alpha = (x_{g_0 g} \oplus x_g^0\ )_{g \in G}$$

where $x_{g_0 g} \oplus x_g^0 = x_{g_0 g} + x_g^0 \pmod 2$. Then these mappings T_α are bijective on X. For the law of composition

$$\alpha\beta = (x,g_0)(y,h_0) = (x^{h_0} + y, g_0 h_0) = r, \text{ where } x^{h_0} = (x_{h_0 g})_{g \in G}\,,$$

$\mathscr{G}$ is a semigroup. Since $\mathscr{G}$ has the identity element $(0,e)$ and the inverse of (x,g) in $\mathscr{G}$ is given by $(x^{g^{-1}},g^{-1})$, it follows that $\mathscr{G}$ is a group. Also it can be shown that $\mathscr{G}$ is a free, ergodic and non-measurable $(X,\mathscr{S},\mu)$-group. Consequently, the corresponding $\mathbf{M}(X,\mathscr{G},\mu)$ of Theorem 7.2 is a type III-factor on the separable space $H_\mu^{\mathscr{G}}$. Pukánsky [28] showed that this factor fails the property (L). For later use, we shall denote this factor by $\mathbb{P}$. (Note that in the study of Pukánsky [28] or in that of Sakai [32], the factor $\mathbb{P}$ is not distinguished for different pairs (p_1,q_1) and (p_2,q_2)).

Since the property (L) is an isomorphism invariant, the above examples imply the following:

THEOREM 11.1. *There exist at least two non-isomorphic type* III*-factors on a separable Hilbert space H, one satisfying the property* (L) *and the other failing it.*

12. A Continuum of Non-Isomorphic Type II_1-Factors

After the publication of "On Rings of Operators IV" in 1943, only two non-isomorphic type II_1-factors were known for many years. In 1963 J. Schwartz [34] introduced as isomorphism invariant called "the property (P)" and using (P) distinguished two non-isomorphic non-hyperfinite type II_1-factors. After the publication of [34], many mathematicians became interested in the construction of new non-isomorphic type II_1-factors. Using the notions of central and hyper-central sequences in a type II_1-factor, Dixmier and Lance constructed two new examples of non-isomorphic type II_1-factors in [10]. New type II_1-factors were also given by Wai-mee-ching [4], Sakai [29] and Zeller-Meir [39]. Thus only nine non-isomorphic type II_1-factors were known before the publication of [17] and [18] by Dusa McDuff.

In this section we briefly sketch some of the ideas used by Dusa McDuff [17, 18] and describe the construction of a continuum of non-isomorphic type

II_1-factors following Sakai [32]. For details of the proof, the reader is referred to Sakai [32, pp. 183–192].

Motivated by the hypothesis in Lemma 6.2.1 of [21] (see Theorem 10.2 above), Dixmier and Lance introduced in [10] the notion of a residual subgroup H of G, according to which the hypothesis in the said lemma of [21] implies that $\{e\}$ is a residual subgroup of the ICC-group G. Since it is not known whether the finite product of residual subgroups is residual, Dusa McDuff defined in [17] a much stronger notion of strongly residual subgroups for which the said propety holds and considered strongly residual sequences of subgroups in G. Using these notions, and proving many technically complex lemmas, she constructed an uncountable family of type II_1-factors in [18].

Let $G_1, G_2, \dots; H_1, H_2, \dots$ be two sequences of groups. We denote by $(G_1, G_2, \dots; H_1, H_2, \dots)$ the group generated by the G_i's and the H_i's with additional relations that H_i, H_j commute elementwise for $i \neq j$ and G_i, H_j commute elementwise for $i \leqslant j$. Let $L_1 = (\mathbb{Z}, \mathbb{Z}, \dots; \mathbb{Z}, \mathbb{Z}, \dots)$. Let L_k be defined inductively by $L_k = (\mathbb{Z}, \mathbb{Z}, \dots; L_{k-1}, L_{k-1}, \dots)$ for $k > 1$.

Let π be a sequence of positive integers. Let $M_n(\pi) = \Sigma_{i=1}^{n} \oplus L_{p_i}$ if $\pi = (p_1, p_2, \dots)$; and $M_n(\pi) = \Sigma_{i=1}^{n} \oplus L_{p_i}$ for $n \leqslant n_0$ and $M_n(\pi) = M_{n_0}(\pi)$ for $n > n_0$, if $\pi = (p_1, \dots, p_{n_0})$. Let $G(\pi) = (\mathbb{Z}, \mathbb{Z}, \dots; M_1(\pi), M_2(\pi), \dots)$. Then one has the following result.

THEOREM 12.1. *If $\pi_1 = (p_i)$ and $\pi_2 = (q_i)$ are two sequences of positive integers such that $\pi_1 \neq \pi_2$ as sets, then $\mathfrak{U}(G(\pi_1))$ and $\mathfrak{U}(G(\pi_2))$ (see Notation 9.1) are non-isomorphic type II_1-factors. None of these factors is hyperfinite. Thus there exists a continuum of non-isomorphic type II_1-factors.*

13. SAKAI'S CONSTRUCTION OF UNCOUNTABLY MANY NON-HYPERFINITE TYPE III AND TYPE II_∞-FACTORS

In the set up of W^*-algebras, Sakai [30, 32] extended the notion of central sequences and, using the type III-factor $\mathbb{P}$ of Section 11 and the ICC-groups $G(\pi)$ of Section 12 above, constructed a continuum of non-isomorphic type III-factors and deduced the existence of a continuum of non-isomorphic type II_∞-factors. This is outlined as follows:

A B^*-algebra W is called a W^*-algebra if there exists a Banach space W_*

such that W is the Banach space dual of W_*. Let W a W^*-algebra in the sequel. The weak*-topology $\sigma(W, W_*)$ is called the σ-topology of W. A *-homomorphim $\Phi : W_1 \longrightarrow W_2$ between two W^*-algebras W_1 and W_2 is called a W^*-homomorphism if it is continuous for the σ-topologies of W_1 and W_2.

Given a W^*-algebra W, there exists a faithful W^*-representation Φ of W into $L(K)$ of some Hilbert space K (K can be finite diemnsional or of arbitrary dimension) such that $\Phi(W)$ is a *-subalgebra closed in the weak operator topology of $L(K)$ (see Section 1.16 of Sakai [32]). In such case, W is said to have a W^*-representation (Φ, K). Moreover, when W contains the identity, then $\Phi(W)$ is a von Neumann algebra on K.

Let $\mathbf{T} = \{\psi : \psi$ a σ-continuous positive linear form on $W\}$. For each $\psi \in \mathbf{T}$, let $\alpha_\psi(x) = (\psi(x^*x))^{1/2}$ for $x \in W$. The locally convex topology defined on W by the family $\{\alpha_\psi : \psi \in \mathbf{T}\}$ of semi-norms is called the s-topology of W. If $\{X_n : n \in \mathbb{N}\}$ is a uniformly bounded sequence in W, we say that $\{X_n\}$ is a *central sequence* if $X_n X - X X_n \longrightarrow 0$ in s-topology for all $X \in W$.

From the theory of tensor products of von Neumann algebras (see [32]) it follows that $\mathbb{P} \otimes \mathbf{M}$ is a factor for any factor $\mathbf{M}$ and is of type III, where $\mathbb{P}$ is as in Example 11.2.

Considering $\mathscr{A}_i = \mathbb{P} \otimes \mathfrak{U}(G(\pi_i))$, $i = 1,2$ as W^*-algebras with identity and assuming them to be isomorphic for two different sequences of positive integers π_1 and π_2 (where $\mathfrak{U}(G(\pi_i))$ are as in Section 12), Sakai [32] arrives at a contradiction after proving many intermediate lemmas, in which the above generalized notion of central sequences plays a key role.

THEOREM 13.1. *Let π_1 and π_2 be two sequences of positive integers which are different as sets. Let $G(\pi_1)$ and $G(\pi_2)$ be the ICC-groups constructed in Section* 12 *above. Then $\mathbb{P} \otimes \mathfrak{U}(G(\pi_1))$ and $\mathbb{P} \otimes \mathfrak{U}(G(\pi_2))$ are non-isomorphic type* III*-factors. Moreover, these factors are non-hyperfinite* (*see Definition* 8.1). *Thus there exists a continuum of non-isomorphic non-hyperfinite type* III*-factors on a separable Hilbert space.*

Note 13.1. In the next section, following Powers [27] we also give the construction of a continuum of non-isomorphic hyperfinite type III-factors.

Since $\mathbb{P}$ is of type III, $\mathbb{P}$ is isomorphic to $\mathbb{P} \otimes L(H)$ for a separable space H. Consequently, we deduce from Theorem 13.1 the following:

THEOREM 13.2. *If H is separable and if π_1, π_2, $G(\pi_1)$ and $G(\pi_2)$ are as in Theorem* 13.1, *then $L(H)\otimes\mathfrak{U}(G(\pi_1))$ and $L(H)\otimes\mathfrak{U}(G(\pi_2))$ are non-isomorphic type II_∞-factors. Consequently, there exists a continuum of non-isomorphic type II_∞-factors on a separable Hilbert space.*

For the details of this section the reader is referred to Sakai [32, pp. 193-202].

14. POWERS' CONSTRUCTION OF A CONTINUUM OF NON-ISOMORPHIC HYPERFINITE TYPE III-FACTORS

The construction of Powers [27] is based on the infinite product of a sequence of type I_2-factors, each one being considered as a C^*-algebra with identity. The reader may refer to [12] for details of the construction of infinite tensor products of C^*-algebras.

Suppose that $\mathscr{B}_n = \mathscr{B}$ is a type I_2-factor on a separable Hilbert space H for each $n \in \mathbb{N}$. Let (p_n) be a sequence of positive numbers $0 < p_n < 1/2$. For $\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}$ $(\alpha,\beta,\gamma,\delta$ complex numbers$)$, let

$$\psi_{p_n}\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix} = \alpha p_n + \delta(1-p_n)\,.$$

Then ψ_{p_n} is a state (= positive linear form with $\|\psi_{p_n}\| = 1$) on $\mathscr{B}_n$. Then let $\Psi(p_n) = \otimes_{n=1}^{\infty}\psi_{p_n}$ be the infinite product state of $(\psi_{p_n})_1^\infty$ on $\otimes_{n=1}^{\infty}\mathscr{B}_n$ (see Section 1.23, Chapter 1 of Sakai [32]).

It is known that the state $\Psi(p_n)$ induces a $*$-representation $\Pi_{\Psi(p_n)}$ of $\mathscr{A} = \otimes_{n=1}^{\infty}\mathscr{B}_n$ on a Hilbert space $H_{\Psi(p_n)}$ (see p. 40 of [32]). The von Neumann algebra $\mathscr{R} = (\pi_{\Psi(p_n)}(\mathscr{A}))''$ is called the W^*-infinite tensor product of $(\mathscr{B}_n)_1^\infty$ by the infinite product state $\Psi(p_n)$. In this particular case, $\mathscr{R}$ is a factor.

If there exists a positive number δ with $\delta < p_n < 1/2 - \delta$ for each n, then it can be shown that $\mathscr{R} = (\pi_{\Psi(p_n)}(\mathscr{A}))''$ is a type III-factor and that $\mathscr{R}$ is spatially isomorphic to the factor $\mathscr{M}_{(p_n)_1^\infty}$ of Example 11.1. (See p. 206 of [32]).

When we take $p_n = \lambda$ for all n with $0 < \lambda < 1/2$, the associated type III-factor $\mathscr{M}_{(p_n)_1^\infty}$ is denoted by $\mathscr{M}_\lambda$ and is called the *Powers factor* of λ.

Introducing an isomorphism invariant called "the property L_λ", Powers [27] obtained the following:

THEOREM 14.1. *For $\lambda_1,\lambda_2 \in (0,1/2)$ with $\lambda_1 \neq \lambda_2$, their Powers factors $\mathscr{M}_{\lambda_1}$ and $\mathscr{M}_{\lambda_2}$ are non-isomorphic hyperfinite type* III*-factors. Consequently,* *there exists an uncountable family of non-isomorphic hyperfinite type* III*-factors on a separable Hilbert space.*

The reader should note the difference between Theorems 13.1 and 14.1.

15. ITPFI-FACTORS

In [24] von Neumann observed that certain type III-factors could be obtained as an infinite tensor product of finite type I-factors. But no proof of his statement was given in any of his publications. It was only in 1963 that Bures [3] gave the proof of the above assertion along with a partial type classification of these infinite products. Such infinite tensor products of finite type I-factors are themselves factors and are called ITPFI-factors.

In Section 14 we saw that the Powers factors $\mathscr{M}_\lambda$ are ITPFI-factors of special type, with the constituent factors of type I_2. Analysing the work of Powers [27], Araki and Woods studied in [2] the complete type classification of general ITPFI-factors by introducing the isomorphism invariants r_∞ and ρ. Without going into a detailed definition of an ITPFI-factor $\mathbf{M}$, let us simply mention some of the principal results of Araki and Woods [2], reformulated in a form comparable with the later results of Connes (see the next section).

Let us denote the Powers factor $\mathscr{M}_\lambda$ by R_x, where $\lambda = x/(1+x)$ so that $x \in (0,1)$ as λ varies in $(0,1/2)$. We define R_0 as the type I_∞-factor and R_1 as the hyperfinite type II_1-factor on a separable Hilbert space H. (Note that these are unique up to isomorphism). The *asymptotic ratio set* $r_\infty(\mathbf{M})$ for an ITPFI-factor $\mathbf{M}$ defined in terms of the eigen values sets corresponding to the tracial states of the constituent factors is shown in [2] to be the same as the set $\{0 \leqslant x < \infty : \mathbf{M} \sim \mathbf{M} \otimes R_{f(x)}\}$, where "$\sim$" denotes "isomorphic" and $f(x) = x$ for $0 \leqslant x \leqslant 1$ and $f(x) = x^{-1}$ for $1 < x < \infty$. This result suggested the definition of $r_\infty(\mathbf{M}) = \{0 \leqslant x < \infty : \mathbf{M} \sim \mathbf{M} \otimes R_{f(x)}\}$ for an arbitrary factor $\mathbf{M}$.

For two factors R_1 and R_2, it is known that $R_1 \otimes R_2$ is also a factor, which is of type III (resp., of type II) if $\mathscr{R}_1$ or $\mathscr{R}_2$ is of type III (resp,. if one of them is of type II and the other is semi-finite).

Araki and Woods [2] proved that $r_\infty(\mathbf{M})$ is an isomorphism invariant and Araki [1] showed that $r_\infty(\mathbf{M})$ must be one of the sets $\{0\}$, $\{1\}$, $S_0 = \{0,1\}$, $S_x = \{0,1,x^n : n \in \mathbb{Z}\}$, $0 < x < 1$ and $S_1 = [0,\infty)$ for an ITPFI-factor $\mathbf{M}$. (Here the original notation is changed in terms of the invariant S of [6]).

THEOREM 15.1. *Except for the case* S_0, $r_\infty(\mathbf{M}) = r_\infty(\mathbf{N})$ *for two ITPFI-factors* **M** *and* **N** *implies that* **M** *and* **N** *are isomorphic.*

The other isomorphism invariant $\rho(\mathbf{M})$ for an arbitrary factor **M** is given in [2] as below:

$$\rho(\mathbf{M}) = \{0 \leqslant x < \infty : R_{f(x)} \sim R_{f(x)} \otimes \mathbf{M}\}.$$

Using the invariant ρ, the following interesting theorem was obtained in [2].

THEOREM 15.2. *There exists a continuum of non-isomorphic ITPFI-factors in the class* S_0.

It is interesting to observe that all the Powers factors $\mathscr{M}_\lambda$ $(0 < \lambda < 1/2)$ belong to the class S_0, which are already known to be non-isomorphic ITPFI-factors. (see Theorem 14.1.)

Thus for the first time, after the publication of [21], factors given by different constructions were identified. The classification by r_∞ and ρ was generalized later by Krieger [14, 15, 16] to factors constructed from ergodic transformations. For more information on ITPFI-factors the reader is referred to Woods [38].

16. RESULTS OF CONNES [6] AND TAKESAKI [36,37]

Using the Tomita-Takesaki theory of modular Hilbert algebras and the non-commutative integration theory, Connes [6] gave an isomorphism invariant $T(\mathbf{M})$ for an arbitrary von Neumann algebra **M**. Theorem 14.1 above and the non-isomorphism of the non-hyperfinite family $\mathfrak{U}(G) \otimes \mathscr{M}_\lambda$, $0 < \lambda < 1/2$, with G as in Example 10.1, were deduced from the following theorem.

THEOREM 16.1. *If* **M** *is an ITPFI-factor, then* $T_0 \in T(\mathbf{M})$ *if and only if* $\exp(-2\pi / T_0) \in \rho(\mathbf{M})$, *where* $\rho(\mathbf{M})$ *is the invariant given by Araki and Woods in* [2]. (*See Section* 15.)

Another interesting result about $T(\mathbf{M})$ given in [6] is the following:

THEOREM 16.2. *Every subgroup* G *of* $\mathbb{R}$ *is the set* $T(\mathbf{M})$ *of a countably decomposable factor* **M**. *When* G *is countably infinite,* **M** *is a factor on a separable Hilbert space. Futhermore, there exists a countably decomposable type* III*-factor* **M** *such that* $T(\mathbf{M}) = \mathbb{R}$.

In [6] Connes gave another isomorphism invariant $S(\mathbf{M})$ for a factor **M** and showed that $\mathbf{M}$ is semi-finite if and only if $S(\mathbf{M}) = \{1\}$. He also proved that the invariant $T(\mathbf{M})$ doesn't determine $S(\mathbf{M})$, in the sense that two factors $\mathbf{M}_1$ and $\mathbf{M}_2$ with $T(\mathbf{M}_1) \neq T(\mathbf{M}_2)$ can have $S(\mathbf{M}_1) = S(\mathbf{M}_2)$.

THEOREM 16.3. *For an ITPFI-factor* $\mathbf{M}$ *of type* III, $S(\mathbf{M}) = r_\infty(\mathbf{M})$, *where* $r_\infty(\mathbf{M})$ *is the asymptotic ratio set of* $\mathbf{M}$ *(see Section* 15*).*

Connes [6] also gave an example of a non-ITPFI-factor $\mathbf{M}$ for which $S(\mathbf{M}) \neq r_\infty(\mathbf{M})$. Also in [6] was given a non-hyperfinite ITPFI-factor, contrary to the factors $\mathcal{M}_\lambda$ of Powers.

The most important results of Connes [6] are those which characterize type III-factors. In this direction, he introduced the following:

DEFINITION 16.1. Let $\mathbf{M}$ be a factor and $\lambda \in [0,1]$. $\mathbf{M}$ is said to be of type III_λ if

$$S(\mathbf{M}) = \{0, 1, \lambda^n : n \in \mathbb{Z}\} \quad \text{for} \quad 0 < \lambda < 1,$$

$$S(\mathbf{M}) = \{0,1\} \quad \text{for} \quad \lambda = 0 \qquad \text{and} \qquad S(\mathbf{M}) = [0,\infty) \quad \text{for} \quad \lambda = 1.$$

Since $0 \in S(\mathbf{M})$ for $\lambda \in [0,1]$, it follows that every type III_λ-factor is necesarily of type III. Connes [6] proved the following result in the reverse direction.

THEOREM 16.4. *For every countably decomposable factor* $\mathbf{M}$ *of type* III *there corresponds a unique* $\lambda \in [0,1]$ *such that* $\mathbf{M}$ *is of type* III_λ *so that every type* III*-factor* $\mathbf{M}$ *on a separable Hilbert space is of type* III_λ *for some* $\lambda \in [0,1]$.

He also gave the following theorem of characterization of type III_λ-factors for $\lambda \in [0,1)$.

THEOREM 16.5. (i) *All factors* $\mathbf{M}$ *of type* III_λ *for* $\lambda \in (0,1)$ *can be realized as the crossed product of a type* II_∞*-factor* $\mathfrak{N}$ *by a suitable automorphism* Θ *of* $\mathfrak{N}$. (ii) *A factor* $\mathbf{M}$ *of type* III_0 *is the crossed product of a von Neumann algebra* $\mathfrak{N}$ *of type* II_∞ *with nonatomic centre by a trace diminishing automorphism* Θ *of* $\mathfrak{N}$ *which is ergodic on the centre of* $\mathfrak{N}$.

It is known from [13] that a result similar to (i) and (ii) above doesn't hold for type III_1-factors.

The work of Connes [6] has many other interesting results, which we omit here for lack of space. Moreover, Theorem 16.5 is a remarkable achievement in the classification theory of type III-factors and the work of Connes [6] is so important and original that it earned him the Fields medal of that decade.

Finally, we include the structure theorem of arbitrary type III-von Neumann algebras obtained by Takesaki [36] independent of Connes [6].

THEOREM 16.6. *A von Neumann algebra* $\mathfrak{R}$ *of type* III *is uniquely expressable as the crossed product of a von Neumann algebra* $\mathfrak{R}_0$ *of type* II_∞ *by a one-parameter automorphism group which leaves a trace of* $\mathfrak{R}_0$ *relatively invariant, but not invariant.*

For details of this section, the reader is referred to [6], [36] and [37].

Finally we observe that so far no structure theory of type II_1-factors is known, even though distinct uncountable families of non-isomorphic type II_1-factors have been constructed by different authors. See [5, 18, 31].

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