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Miščenko's Theorem for Bitopological Spaces

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In 1945, V.E. Šneider proved that every Hausdorff compact space with a G_δ–diagonal is metrizable [12]. By using Šneider's theorem, M. Katětov stated in 1948 that a Hausdorff compact space (X,T) is metrizable if and only if the space $(X\times X\times X, T\times T\times T)$ is hereditarily normal [4] and A. Miščenko proved in 1962 his celebrated theorem that a Hausdorff compact space with a point–countable base is metrizable [8].

Bitopological extensions of the theorems of Šneider and Katětov have been obtained in [9] and [10]. We here obtain the following generalization of Miščenko's theorem: A pairwise Hausdorff pairwise countably compact bitopological space (X,P,Q) is quasi–metrizable if and only if both P and Q have a point–countable base. Actually, we will prove a more general result in terms of point–countable T_1–separating open covers. (Further results about pairwise compact quasi–metrizable spaces may be found in [6] and [7]).

Let us recall some definitions.

A quasi–metric on a set X is a non–negative real–valued function d on $X\times X$ such that for all $x,y,z\in X$, i) $d(x,y)=0 \Leftrightarrow x=y$; ii) $d(x,y)\leqslant d(x,z)+d(z,y)$.

Each quasi–metric d on X induces a T_1 topology $T(d)$ on X which has as a base the family $\{B_d(x,r) : x\in X,\ r>0\}$ where $B_d(x,r)=\{y\in X : d(x,y)<r\}$.

Note that if d is a quasi–metric on X, then the function d^{-1}, defined by $d^{-1}(x,y)=d(y,x)$ for each $x,y\in X$, is also a quasi–metric on X called conjugate of d.

Bitopological spaces appear in a natural way when one considers the topologies $T(d)$ and $T(d^{-1})$ naturally associated with a quasi–metric and its

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conjugate. A bitopological space is [5] and ordered triple (X,P,Q) such that X is a nonempty set and P and Q are topologies on X.

A bitopological space (X,P,Q) is called:

i) pairwise regular if for all $x \in X$, the Q-closed P-neighborhoods of x form a base for the P-neighborhoods of x and P-closed Q-neighborhoods of x form a base for the Q-neighborhoods of x [5].

ii) pairwise Hausdorff if for $x \neq y$ there is a P-neigborhood of x and a disjoint Q-neigborhood of y.

iii) quasi-metrizable if there is a quasi-metric d on X such that $T(d) = P$ and $T(d^{-1}) = Q$.

In [2], P. Fletcher, H.B. Hoyle III and C.W. Patty introduced the notion of a pairwise compact bitopological space and proved that every pairwise Hausdorff pairwise compact space is pairwise regular. In the following we will make use of a useful characterization of pairwise (countably) compact spaces due to M.K. Singal and R. Singal [11] (see also [1]):

A bitopological space (X,P,Q) is pairwise (countably) compact if and only if every proper P-closed subset is Q-(countably) compact and every proper Q-closed subset is P-(countably) compact.

The letter $\mathbb{N}$ will denote the set of positive integers. If P is a topology for a set X and if $A \subseteq X$, we write $cl_P A$ for the closure of A in (X,P).

We will also use the following auxiliary results:

LEMMA 1. ([9]) *A pairwise countably compact space (X,P,Q) such that each proper countably compact subspace of (X,P) has a G_δ-diagonal is pairwise compact.*

LEMMA 2. ([9]) *Let (X,P,Q) be a pairwise Hausdorff pairwise countably compact space and let P be second countable. Then (X,P,Q) is quasi-metrizable and Q is second countable.*

Recall that a topological space (X,T) has countable pseudo-character if for each $x \in X$, $\{x\} = \cap\{V_n(x) : n \in \mathbb{N}\}$ where each $V_n(x)$ is an open set.

LEMMA 3. *Let (X,P,Q) be a pairwise Hausdorff pairwise countably compact space such that (X,P) has a G_δ-diagonal and (X,Q) has countable pseudo-character. Then (X,P,Q) is quasi-metrizable.*

Proof. By Lemma 1, (X,P,Q) is pairwise compact. Fix $x \in X$. Then $\{x\} =$

$\cap\{V_n(x): n \in \mathbb{N}\}$ where each $V_n(x)$ is Q–open. So $X \setminus \{x\} = \cup\{X \setminus V_n(x): n \in \mathbb{N}\}$. Take a point in X, $y \neq x$, and let $\{y\} = \cap\{V_n(y): n \in \mathbb{N}\}$ where each $V_n(y)$ is Q–open. Then there is a $k \in \mathbb{N}$ such that $x \in X \setminus V_k(y)$. Thus $X = (X \setminus V_k(y)) \setminus (\cup\{X \setminus V_n(x): n \in \mathbb{N}\})$. Now $X \setminus V_k(y)$ is P–compact since it is a proper Q–closed subset of X. Similarly, each $X \setminus V_n(x)$ is P–compact. Therefore (X,P) is σ–compact and, consequently, it is a Lindelöf space. The quasi–metrizability of (X,P,Q) follows from the fact [9, Theorem 3] that a pairwise Hausdorff pairwise countably compact space (X,P,Q) is quasi–metrizable if (X,P) is a Lindelöf space with a G_δ–diagonal. ∎

A cover $\mathscr{C}$ of a set X is called T_1–separating if for each $x \in X$, $\{x\} = \cap\{C \in \mathscr{C} : x \in C \in \mathscr{C}\}$ [3].

THEOREM. *A pairwise Hausdorff pairwise countably compact space (X,P,Q) is quasi–metrizable if and only if (X,P) has a point–countable T_1–separating open cover and (X,Q) has countable pseudo–character.*

Proof. Sufficient condition: Let $\mathscr{C}$ be a point–countable T_1–separating open cover for (X,P). For each P–countably compact subset F of X put

$$\mathscr{C} \cap F = \{C \cap F : C \in \mathscr{C}\}.$$

Then $\mathscr{C} \cap F$ is a point–countable T_1–separating open cover of the subspace $(F,P|F)$. Since a countably compact topological space has a G_δ–diagonal if and only if it has a point–countable T_1–separating open cover [3, p. 475], $(F,P|F)$ has a G_δ–diagonal. By Lemma 1, (X,P,Q) is pairwise compact.

Now fix $x \in X$. Then $\{x\} = \cap\{V_n(x): n \in \mathbb{N}\}$ where each $V_n(x)$ is Q–open. Since (X,P,Q) is pairwise regular, for each $n \in \mathbb{N}$ there is a Q–open set $W_n(x)$ such that $x \in W_n(x) \subseteq cl_P\, W_n(x) \subseteq V_n(x)$. Thus $\{x\} = \cap\{cl_P\, W_n(x): n \in \mathbb{N}\}$. But $X \setminus W_n(x)$ is P–countably compact, so that the subspace

$$(X \setminus W_n(x), P|X \setminus W_n(x))$$

has a G_δ–diagonal as we have shown above. By Lemma 3, the subspace

$$(X \setminus W_n(x), P|X \setminus W_n(x), Q|X \setminus W_n(x))$$

is quasi–metrizable and by [9, Theorem 1], $P|X \setminus W_n(x)$ is second countable. Then $P|X \setminus cl_P\, W_n(x)$ is also second countable. Now take a point in X, $y \neq x$; thus $\{y\} = \cap\{cl_P\, W_n(y) : n \in \mathbb{N}\}$ where each $W_n(y)$ is Q–open, so that there is a $k \in \mathbb{N}$ such that $x \in X \setminus cl_P\, W_k(y)$. Since $P|X \setminus cl_P\, W_k(y)$ is second countable, P is second countable. By Lemma 2, (X,P,Q) is quasi–metrizable.

Necessary condition: It follows immediately from the fact that if (X,P,Q) is a pairwise Hausdorff pairwise countably compact quasi–metrizable space then both P and Q are second countable [9, Theorem 1]. ∎

COROLLARY. *A pairwise Hausdorff pairwise countably compact space (X,P,Q) is quasi–metrizable if and only if both (X,P) and (X,Q) have a point–countable base.*

Proof. If both (X,P) and (X,Q) have a point–countable base, then (X,P) has a point–countable T_1–separating open cover and (X,Q) has countable pseudo–character. ∎

Note that if in the above corollary we put $P=Q$ one obtains Miščenko's theorem.

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