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Non-attainable Boundary Values of H^∞ Functions

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1. STATEMENT OF RESULT

Let B_d denote the unit ball in C^d, and let $H^\infty(B_d)$ be the space of bounded analytic functions in B_d. It is well-known that if m is in $H^\infty(B_d)$, then $m^*(\zeta) := \lim_{r\to 1^-} m(r\zeta)$ exists for σ_d-a.e. ζ on the boundary of B_d, where σ_d is normalized $(2d-1)$-dimensional Lebesgue measure on the sphere S_d [5]. Moreover, given m^*, one can recover m by integrating against the Poisson kernel. We are interested in the following question: if g is a non-negative function in $L^\infty(\sigma_d)$, when does there exist a function m in $H^\infty(B_d)$ with $|m^*| = g$ σ_d-a.e.?

As the function $\log|m|$ is subharmonic, there is one obvious necessary condition, namely that

$$\int_{S_d} \log(g)\, d\sigma_d > -\infty. \tag{1.1}$$

For $d=1$, (1.1) is also sufficient, as g is then the modulus of the outher function

$$g(z) = \exp\left[\int_{S_d} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log(e^{i\theta})\, d\sigma_1(e^{i\theta})\right].$$

(This answer to our question for $d=1$ is due to Szegö [7]).

For $d>1$, condition (1.1) is necessary and sufficient for g to be the modulus of a function in the larger Nevalinna class $N(B_d)$, consisting of those holomorphic functions f on the ball for which

$$\sup_{0<r<1} \int_{S_d} \log^+ |f(r\zeta)|\, d\sigma_d(\zeta) < \infty$$

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[6, Theorem 10.11]. It is no longer sufficient, however, for g to be the modulus of a bounded analytic function, because the function

$$\zeta \longrightarrow \operatorname{ess} \sup_{-\pi \leqslant \theta \leqslant \pi} |m^*(e^{i\theta}\zeta)|$$

must be lower semi-continuous on S_d if m is in $H^\infty(B_d)$ [6]. If g is bounded below, then (1.1) and some semi-continuity condition is sufficient: in [6, Theorem 12.5], Rudin proves that if g is log-integrable, and there exists some non-zero f in $H^\infty(B_d)$ with $g \geqslant |f^*|$ a.e. and $g/|f^*|$ lower semi-continuous, then there does exist m in $H^\infty(B_d)$ with $g = |m^*|$ a.e.

Our result is that if g is not bounded below, then log-integrability and continuity do not suffice for g to be the modulus of a holomorphic function:

THEOREM 1.2. *Let $d \geqslant 2$. There is a continuous non-negative function g on S_d, vanishing only at the point e_1, and satisfying $\int_{S_d} \log(g) d\sigma_d > -\infty$, with the property that the only function m in $H^\infty(B_d)$ with $|m^*| \leqslant g$ almost everywhere $[\sigma_d]$ is the zero function.*

2. IDEA OF PROOF

The Smirnov class $N^+(B_d)$ consists of those functions f in $N(B_d)$ for which $\{\log^+|f(r.)| : 0 < r < 1\}$ is a uniformly integrable family on S_d. Equipped with the metric $\rho(f,g) = \int_{S_d} \log(1 + |f - g|) d\sigma_d$, it becomes a topological vector space that is not locally convex. Just as in the $d = 1$ case [3], it can be realised as an inductive limit of Hilbert spaces. For w a non-negative function in $L^1(\sigma_d)$, let $P^2(w\sigma_d)$ denote the closure of the polynomials in $L^2(w\sigma_d)$. Then

$$N^+(B_d) = \cup_{\log(w) \in L^1(\sigma)} P^2(w\sigma_d).$$

$N^+(B_d)$ is, however, strictly larger than

$$\cup_{m \in H^\infty(B_d)} P^2(|m^*|^2 \sigma_d) \tag{2.1}$$

The idea is to construct a linear functional Γ on (2.1) that does not extend to $N^+(B_d)$, and show that there is actually a continuous w such that Γ is not bounded on $P^2(w\sigma_d)$.

Now if $\Gamma(\zeta^\alpha) = c_\alpha$, then Γ is bounded on $P^2(|m^*|^2\sigma_d)$ if and only if the function $f(z) = \Sigma\, c_\alpha z^\alpha$ is in range of the co-analytic Toeplitz operator $T_{\overline{m}}$ on $H^2(\sigma_d)$ [1]. (For any measure μ, the Toeplitz operator $T_{\overline{m}}$ is defined on $P^2(\mu)$

by

$$T_{\overline{m}}^{P^2(\mu)} f = P\overline{m}f,$$

where P is the orthogonal projection from $L^2(\mu)$ onto $P^2(\mu)$.)

We prove the following:

THEOREM 2.2. *Let* $f(z_1,\dots,z_d) = f_1(z_1) = \Sigma_{n=0}^{\infty} a_n z_1^{\,n}$, *and suppose that* $a_n = O(e^{-cn^{1/2+\epsilon}})$ *for some* $\epsilon, c > 0$. *Then* f *is in the range of the Toeplitz operator* $T_{\overline{m}}^{H^2(B_d)}$ *for every non-zero* m *in* $H^\infty(B_d)$.

The proof of 2.2 uses the techniques of [2] for studying Toeplitz operators on weighted Bergman spaces.

Nawrocki proved in [4] that Γ is in the dual of $N^+(B_d)$ if and only if

$$\Gamma(\zeta^\alpha) = O(e^{-c|\alpha|^{d/(d+1)}} \tag{2.3}$$

We exploit the gap between (2.3) and Theorem 2.2 as follows. Choose some number between $1/2$ and $d/d+1$, e.g. $7/13$, and let Γ be given by

$$\Gamma(z_1^{\alpha_1} \dots z_d^{\alpha_d}) = \delta_{\alpha_2,0} \cdots \delta_{\alpha_d,0}\, e^{\alpha_1^{\,-7/13}}.$$

Let

$$F_{c,w}(z) = \exp\left[c\frac{1-|w|^2}{(1-\langle z, w\rangle)^{d+1}}\right] \tag{2.4}$$

Then, using estimates of Nawrocki on the Taylor coefficients of 2.4 [4], we can pick c_n and r_n so that

$$\sup_{\{\zeta:|\zeta-e_1|\geqslant 1/n\}} c_n|F_{c_n, r_n e_1}(\zeta)| \leqslant 1/2^n,$$

and

$$\int_{S_d} \log(1+|c_n F_{c_n, r_n e_1}| d\sigma_d \leqslant 1/2^n,$$

while $\Gamma(c_n F_{c_n, r_n e_1})$ tends to infinity. Then

$$g(\zeta) = \left[\frac{1}{1+\sum_{n=1}^{\infty}|c_n F_{c_n, r_n e_1}(\zeta)|^2}\right]^{1/2}$$

satisfies the hypothesis of Theorem 1.2.

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