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A Non–Deterministic Time Hierarchy over the Reals

F. CUCKER[1], J.L. MONTAÑA[2] AND L.M. PARDO[2]

[1] *Dept. L.S.I. Facultat d'Informatica, Univ. Politecnica de Catalunya,* 08028 *Barcelona, Spain*
[2] *Dpto. de Matemáticas, Estadística y Computación, Facultad Ciencias, Univ. Cantabria,* 39071 *Santander, Spain*

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0. INTRODUCTION

In their 1989 paper [1], L. Blum, M. Shub and S. Smale introduced a model of computation and a theory of recursiveness that accepted an ordered field or ring as alphabet for the space of admissible inputs: the real Turing machine model. A special emphasis was made in the field of the real numbers, $\mathbb{R}$. This work also made an attempt, under the structural approach to complexity, at a classification of the procedures developed in numerical analysis and a computational geometry involving real numbers as inputs. In particular, analogues of the P and NP classes were introduced there.

In [2] an algebro–geometric characterization of the complexity classes over the reals and a deterministic time hierarchy is stated. The present note is a continuation of the results in [2]. A non–deterministic time hierarchy, as the one in the boolean case, is stated and as a consequence we conclude a feature which is still open in the boolean case: there is a language accepted by a deterministic real Turing machine is simply exponential time which can not be solved in polynomial non–deterministic time.

1. GROUND TOOLS

Notations along these pages will follow essentially those introduced in [1]. By $\mathbb{R}^\infty$ we shall denote the direct sum $\oplus_{i\in\mathbb{N}}\mathbb{R}$, i.e. the set of all sequences of real numbers with only finitely many non–zero coordinates. The size of an element

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$X \in \mathbb{R}^\infty$ is the non-negative integer $|x| = n \in \mathbb{N}$ such that $x_n \neq 0$ is the last non-zero coordinate of x. Also, in analogy with the Boolean case, we shall call "language" any set included in $\mathbb{R}^\infty$ and the language recognized by a Turing machine M over $\mathbb{R}$ will be denoted by $\mathcal{L}(M)$. Moreover, languages recognized by some machine will be called recursively enumerable (and denoted r.e. for short) and r.e. language whose complement is also r.e. will be called recursive. For a language $L \subseteq \mathbb{R}^\infty$, we shall denote by L_n the set of words in L of size $n \in \mathbb{N}$, i.e. $L_n = \{x \in L \,/\, |x| = n\}$.

The running time $R_M(x)$ of a machine M for $x \in \mathcal{L}(M)$ is the number of nodes in M traversed by the computation of M on x until it reaches the output node.

Using these notations, we can define the halting time function associated to a machine M to be

$$\begin{aligned} T_M : \mathbb{N} &\longrightarrow \mathbb{N} \cup \{+\infty\} \\ n &\longmapsto \sup\{R_M(x) \,/\, x \in \mathcal{L}(M),\, |x| = n\} \end{aligned}$$

DEFINITION 1.1. For a total mapping $s: \mathbb{N} \longrightarrow \mathbb{N}$ we define the complexity class $DTIME_{\mathbb{R}}(s)$ to be $\{L \,/\,$there is a machine M, such that, $L = \mathcal{L}(M)$ and $T_M(n) \in O(s)\}$.

DEFINITION 1.2. A machine M is said to be a time bounded computation machine if $T_M(n) < \infty$, for all $n \in \mathbb{N}$. We shall say that a language is time bounded recursive ($L \in TB$ for short) iff L is recognized by a time bounded computation machine.

DEFINITION 1.3. A semialgebraic set S is a subset of some finitely dimensional real affine space, $S \subseteq \mathbb{R}^n$, for $n < \infty$, such that there is a finite polynomials $f_{ij} \in \mathbb{R}[X_1,\ldots,X_n]$ and sign conditions $\epsilon_{ij} \in \{>0\,,=0\,,<0\}$ such that the following equality holds:

$$S = \bigcup_{j=1}^{r} (\cap_{i=1}^{t} \{(x_1,\ldots,x_n) \in \mathbb{R}^n \,/\, \mathrm{sign}\,(f_{ij}(x_1,\ldots,x_n)) = \epsilon_{ij}\}).$$

DEFINITION 1.4. (cf. [2]) A language L is said to be finitely presented if there is a finitely generated field extension $\mathbb{Q}(S)$ of $\mathbb{Q}$, such that for every $n \in \mathbb{N}$, L_n is a semialgebraic subset of $\mathbb{R}^n$ defined over $\mathbb{Q}(S)$.

THEOREM 1.5. *For any language L, L is in TB iff L is finitely presented.*

Proof. See [2]. ∎

The following lemma is taken from [2].

LEMMA 1.6. *Let M be a time bounded machine and L its accepted language. Then, we have $T_M(n) \in \Omega(log_2 \beta_0(L_n))$, where $\beta_0(L_n)$ denotes the number of connected components of L_n.*

2. NON–DETERMINISTIC TIME HIERARCHY

DEFINITION 2.1. For a time bound $s: \mathbb{N} \longrightarrow \mathbb{N}$, we define the class $NTIME_{\mathbb{R}}(s)$ of all those languages such that there is a real Turing machine M accepting inputs from $\mathbb{R}^\infty \times \mathbb{R}^\infty$ and a constant $K \in \mathbb{N}$ satisfying:

For every $x \in \mathbb{R}^\infty$, $x \in L$ if and only if there is $y \in \mathbb{R}^\infty$ such that $(x,y) \in \mathcal{L}(M)$, and $R_M(x,y) \leqslant K \cdot s(|x|)$.

Notice that the class of non–deterministic polynomial time over the reals is

$$NP_{\mathbb{R}} = \bigcup_{k \in \mathbb{N}} NTIME_{\mathbb{R}}(n^k).$$

DEFINITION 2.2. A total mapping $s: \mathbb{N} \longrightarrow \mathbb{N}$ is said to be time constructible if there is a machine M such that the mapping φ_M computed by M restricted to $\mathbb{N}$ coincides with s and, there is a constant $K \in \mathbb{R}$, $K > 0$ such that $R_M(m) \leqslant K \cdot s(m)$, for all excepting a finite number of elements $m \in \mathbb{N}$, where $R_M(m)$ denotes the running time of M on input $m \in \mathbb{N}$.

The most common functions used as time bounds in complexity theory (as, for instance, $\log_2^k(n)$, n^k or $2^{k \cdot n}$, with $k \in \mathbb{N}$ fixed) are time constructible over the reals.

THEOREM 2.3. *Let $t, t': \mathbb{N} \longrightarrow \mathbb{N}$ be two time constructible bounds and assume that $t' \in \omega(t)$. Then, $DTIME_{\mathbb{R}}(t')$ contains a language which is not in $NTIME_{\mathbb{R}}(t)$.*

Proof. Let us consider the language L defined as the set of points $x \in \mathbb{R}^\infty$, such that $x_1 + ix_2$ is a $2^{t'(n)}$–th root of the unity in the complex plane. The following machine

input(x)
$n := |x|$
$m := t'(n)$
$z := (x_1 + ix_2)^{2m}$

if $z = 1$ **then** ACCEPT
else REJECT
fi

accepts L in time $O(t'(n))$. In fact, the first asignements takes constant time, and the following two can be done within time $O(t'(n))$.

On the other hand, if $L \in NTIME_{\mathbb{R}}(t)$, there will be a real Turing machine M verifying the conditions of def 2.1. Observe that if $(x,y) \in \mathcal{L}(M)$ and $R_M(x,y) \leqslant K \cdot t(n)$ for some $y \in \mathbb{R}^\infty$, there is a constant $K_1 \geqslant K$ depending only on M such that only the first $K_1 \cdot t(n)$ coordinates of y are reached along the computation on (x,y). This implies that if $\pi_n : \mathbb{R}^n \times \mathbb{R}^{K_1 \cdot t(n)} \longrightarrow \mathbb{R}^n$ is the projection on the first n coordinates, π_n projects $\mathcal{L}(M)_{n+K_1 \cdot t(n)}$ onto L_n.

Since t is time constructible we acn design a new machine M_1 in the following terms:

input(n,x,y)
$m := K_1 \cdot t(|x|)$
if $T_M(x,y) \leqslant m$ **then** ACCEPT **iff** M ACCEPTS (x,y)
else REJECT
fi

This machine M_1 verifies:

$$R_{M_1}(x,y) \leqslant K_1 \cdot t(|x|) \text{ for all } (x,y) \in \mathcal{L}(M_1).$$
$$\pi_n(\mathcal{L}(M_1)_{n+K_1 \cdot t(n)}) = \pi_n(\mathcal{L}(M)_{n+K_1 \cdot t(n)}) = L_n.$$

As projections does not increase connected components we have

$$\beta_0(\mathcal{L}(M_1)_{n+K_1 \cdot t(n)}) \geqslant \beta_0(L_n) = 2^{t'(n)}$$

Finally, from lemma 1.6 we have

$$K \cdot t(n) \in \Omega(\log_2 \beta_0(\mathcal{L}(M)_{n+K_1 \cdot t(n)}))$$

which contradicts the hypothesis $t' \in \omega(t)$. ∎

COROLLARY 2.4. *For a pair of time bounds $t,t' : \mathbb{N} \longrightarrow \mathbb{N}$ verifying the previous conditions, the following diagram summarizes the hierarchies:*

$$\begin{array}{ccc} DTIME_{\mathbb{R}}(t) & \subsetneq & NTIME_{\mathbb{R}}(t) \\ \not\cap & \# & \not\cap \\ DTIME_{\mathbb{R}}(t') & \subsetneq & NTIME_{\mathbb{R}}(t') \end{array}$$

The following statement is still open in the boolean case:

COROLLARY 2.5. *Denoting by $EXTIME_{\mathbf{R}}$ the class of languages accepted deterministically in simply exponential time we have*

$$EXTIME_{\mathbf{R}} \neq NP_{\mathbf{R}}.$$

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