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An Alternate Criterian of Colocalized–Localized Modules

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The aim of this note is to express a module which is localized as well as colocalized in terms of an object of an intersecting subcategory related to static modules.

Throughout this note it is assumed that, $(A,B,M,N,<,>,[\,,],I,J)$, briefly written as (M,N), is a Morita context, in which, A and B are rings with identity, M is a (B,A)–bimodule, N is an (A,B)–bimodule, $<,>: N\otimes_B M \longrightarrow A$ and $[\,,]: M\otimes_A N \longrightarrow B$, are the (A,A)– and (B,B)–bimodule homomorphisms, respectively, such that if we set, $<,>(n\otimes m) = <n,m>$ and $[\,,](m\otimes n) = [m,n]$, then $<n,m>n' = n[m,n']$ and $m<n,m'> = [m,n]m'$, for all $n,n' \in N$ and $m,m' \in M$. Finally, $I = Im<,>$ and $J = Im[\,,]$ are two trace ideals.

The Morita context (M,N) is said to be injective (respect. projective), abbreviated as, "IMC" (respect. "PMC"), if the bimodule homomorphisms, $<,>$ and $[\,,]$, are monomorphisms (respect. epimorphisms).

Keeping the notation of [6] in action, for the ideal T of A, we denote by $\mathscr{C}_T$ and $\mathscr{L}_T$ the full additive subcategories of Mod$-A$ of all colocalized and localized modules, respectively, where:

$$\mathscr{C}_T = \{\, V \in \mathrm{Mod}-A \mid VT = V \text{ and } V \text{ is } T\text{–flat}\},$$

and

$$\mathscr{L}_T = \{\, V \in \mathrm{Mod}-A \mid \mathrm{Ann}_V(T) = 0 \text{ and } V \text{ is } T\text{–injective}\}.$$

In addition to above, we denote by $\mathscr{K}_T$ the full additive subcategory of Mod$-A$ whose objects are the common objects of $\mathscr{C}_T$ and $\mathscr{L}_T$. In other words, $\mathscr{K}_T = \mathscr{C}_T \cap \mathscr{L}_T$.

Let us divert our attention towards the categories of static modules and their related intersecting subcategories.

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Recall from [9] that the intersecting subcategory $\mathscr{X}(A)$ is the full additive subcategory of Mod$-A$ of all those modules which are common objects of Mod(A_M), Mod(A^N), and $\mathscr{D}(A)$, where Mod(A_M) (respect. Mod(A^N)) is the full additive subcategory of Mod$-A$ of all those objects which remain invariant under the composition functor $\mathrm{Hom}_A(M,-)\otimes_B M$ (respect. $\mathrm{Hom}_B(N,-\otimes_A N)$ in a natural way, and $\mathscr{D}(A)$ is the full additive subcategory of Mod$-A$ of all those objects V for which $V\otimes_A N\cong \mathrm{Hom}_A(M,V)$, via the map $\{,\}: v\otimes n \longrightarrow \{v,n\}$, satisfying $\{v,n\}m = v<n,m>$ for all $m\in M$, $n\in N$ and $v\in V$.

In like way $\mathscr{W}(A)$ is defined which is the full additive subcategory of Mod$-A$ whose class of objects is the intersection of the classes of objects of Mod(A^N) and $\mathscr{D}(A)$.

The basic notion about rings, modules, categories, and Morita theory may be referred from [1], [2], or [3]. Literature about localization and colocalization may be seen in [4], [5], [6], and [7], while for the details about Mod(A_M), Mod(A^N), $\mathscr{D}(A)$, $\mathscr{X}(A)$ and $\mathscr{W}(A)$ and various interactions among them one may refer to [8], [9], and [10].

In [10], we have drawn the conclusion that, if (M,N) is an *IMC*, then under certain condition, $\mathscr{W}(A)\approx\mathscr{W}(B)$. In this work we will derive same conclusion about $\mathscr{K}_I$ and $\mathscr{K}_J$.

We begin by proving that $\mathscr{X}(A)$ is a subcategory of $\mathscr{K}_I$, and likewise, $\mathscr{X}(B)$ is a subcategory of $\mathscr{K}_J$.

THEOREM 1. *Let (M,N) be an IMC, then the objects of $\mathscr{X}(A)$ are $I-$localized as well as $I-$colocalized. In other words $\mathscr{X}(A)$ is a subcategory of $\mathscr{K}_I$.*

Proof. Let K be an object of $\mathscr{X}(A)$. The we get the following sequence of canonical isomorphisms,

$$\begin{aligned} K &\cong \mathrm{Hom}_B(N,K\otimes_A N) && (1a)\\ &\cong \mathrm{Hom}_B(N,\mathrm{Hom}_A(M,K)) && (1b)\\ &\cong \mathrm{Hom}_A(N\otimes_B M,K) && (1c)\\ &\cong \mathrm{Hom}_A(I,K)\ , && (1d)\end{aligned}$$

where the first two isomodphisms, (1a) and (1b), hold because of the fact that K is an object of Mod(A^N) and $\mathscr{D}(A)$, the third isomorphism (1c) is due to the adjoint associativity theorem, and the last one (1d) is satisfied because (M,N) is an *IMC.*

Theorem 1.3 of [6] due to Kato and Ohtake reveals that K is I–localized.

Now we get another sequence of canonical isomorphisms, namely,

$$K \otimes_A I \cong K \otimes_A N \otimes_B M \tag{1e}$$
$$\cong \operatorname{Hom}_A(M,K) \otimes_B M \tag{1f}$$
$$\cong K, \tag{1g}$$

where the isomorphism (1e) holds due to the fact that (M,N) is an *IMC,* and the isomorphisms (1f) and (1g) are satisfied because K is an object of $\mathscr{D}(A)$ and $\operatorname{Mod}(A_M)$ both. From Theorems 2.3 and 2.5 of [6], again due to Kato and Ohtake, one may deduced that K is I–colocalized.

From above two verifications it is concluded that K is an object of $\mathscr{K}_I$. Hence the theorem is proved. The second part can similarly be proved. ∎

The following chain is an obvious notational output of the full subcategories involved in it.

COROLLARY 2. *Let (M,N) be an IMC, then*

$$0 \leqslant \mathscr{X}(A) \leqslant \mathscr{K}_I \begin{matrix} \leqslant \mathscr{C}_I \leqslant \\ \\ \leqslant \mathscr{L}_I \leqslant \end{matrix} \operatorname{Mod}-A.$$

From Theorem 5.1 of [9], Corollary 2.4 of [10], and above result, one may deduce the following equalities of categories.

COROLLARY 3. *If the Morita context (M,N) is a PMC, or equivalently, if A is an object of $\mathscr{X}(A)$ and B is an object of $\mathscr{X}(B)$, then*

$$\mathscr{X}(A) = \mathscr{W}(A) = \mathscr{K}_I = \mathscr{C}_I = \mathscr{L}_I = \operatorname{Mod}-A$$

and

$$\mathscr{X}(B) = \mathscr{W}(B) = \mathscr{K}_J = \mathscr{C}_J = \mathscr{L}_J = \operatorname{Mod}-B.$$

Parallel to Theorem 2.7 of [10] we prove the following.

COROLLARY 4. *Let (M,N) be an IMC, then*

$$\mathscr{K}_I \cap \mathscr{D}(A) = \mathscr{X}(A) \quad \text{and} \quad \mathscr{K}_J \cap \mathscr{D}(B) = \mathscr{X}(B).$$

Proof. We only establish that $\mathscr{K}_I \cap \mathscr{D}(A) = \mathscr{X}(A)$. The second half is its obvious symmetric notational output.

Since $\mathscr{D}(A)$ is common in both, $\mathscr{K}_I \cap \mathscr{D}(A)$ and $\mathscr{X}(A)$, the inclusion

$$\mathscr{X}(A) \leqslant \mathscr{K}_I \cap \mathscr{D}(A)$$

is already proved in Theorem 1. For the given *IMC*, the verification of

$$\mathscr{K}_I \cap \mathscr{D}(A) \leqslant \mathscr{X}(A)$$

may be done by rearranging the terms of the sequences of the canonical isomorphisms (1a) to (1d) and (1e) to (1g). ∎

Theorem 3.2 of [9] states that, "$\mathscr{X}(A)$ and $\mathscr{X}(B)$ are equivalent under the restrictions of functors $\mathrm{Hom}_A(M,-)$ and $\mathrm{Hom}_B(N,-)$".

Finally, using this theorem and above result, it is concluded that.

COROLLARY 5. *Let (M,N) be an IMC. If every I−colocalized−localized module is an object of* $\mathscr{D}(A)$ *and if every J−colocalized−localized module is an object of* $\mathscr{D}(B)$, *then*

$$\mathscr{K}_I = \mathscr{X}(A) \quad \textit{and} \quad \mathscr{K}_J = \mathscr{X}(B).$$

Hence under these considerations,

$$\mathscr{K}_I \approx \mathscr{K}_J.$$

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