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Uniform Density and m–Density for Subrings of $C(X)$

M.I. GARRIDO AND F. MONTALVO

Dpto. de Matemáticas, Universidad de Extremadura, 06071 *Badajoz, Spain*

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For a completely regular space X, $C(X)$ and $C^*(X)$ denote, respectively, the algebra of all real–valued continuous, and continuous and bounded, functions over X. We are interested in the following problem: Is every u–dense subring of $C(X)$ m–dense too?

Recall that the u–topology is defined on $C(X)$ by taking as neighborhood base of $f \in C(X)$ the sets of the form $\{g \in C(X) : |f(x) - g(x)| < \epsilon$ for all $x \in X\}$ where ϵ is a positive real number, and that the m–topology is defined by taking the sets of the form $\{g \in C(X) : |f(x) - g(x)| < u(x)$ for all $x \in X\}$ where u is a positive unit of $C(X)$.

Obviously the m–topology is finer than the u–topology, and it is well–known that the two coincide if and only if X is a pseudocompact space (Hewitt [5]), namely when $C^*(X) = C(X)$. Although, in general, these topologies are different, many families in $C(X)$ that are u–dense are m–dense too. For instance, it was essentially proved by Kurzweil in [6] that the u–density and the m–density are equivalent for the subrings of $C(X)$ that are closed under bounded inversion.

In this note, based on the results obtained in [3], we shall prove that an analogue of Kurzweil's result is not possible for arbitrary subrings of $C(X)$.

We start by setting out a sufficient and necessary condition for the u–dense subrings of $C(X)$ to be m–dense that will be very useful to prove most of the results contained in this paper.

PROPOSITION 1. *Let $\mathfrak{F}$ be a subring of $C(X)$. Then $\mathfrak{F}$ is* m*–dense if and only if it fulfills the following conditions:*

(i) *$\mathfrak{F}$ is* u*–dense.*

(ii) *For each $f \in C(X)$ with $f(x) > 0$ for every $x \in X$, there exists $g \in \mathfrak{F}$ such that $0 < g(x) \leqslant f(x)$ for every $x \in X$.*

COROLLARY 2. (Kurzweil [6]) *Let* $\mathfrak{F}$ *be a subring of* $C(X)$ *closed under bounded inversion (that is, if* $f \in \mathfrak{F}$ *with* $f(x) \geqslant 1$ *for every* $x \in X$, *then* $1/f \in \mathfrak{F}$). *Then* $\mathfrak{F}$ *is* u–*dense if and only if* $\mathfrak{F}$ *is* m–*dense.*

The following example will be the key to establishing our main result.

EXAMPLE 3. Let $\mathfrak{F}$ be the subset of $C(\mathbb{N})$ defined by $\mathfrak{F} = \{(q \cdot z_n)_{n \in \mathbb{N}} : q \in \mathbb{Q}$ and $z_n \in \mathbb{Z}$ for every $n \in \mathbb{N}\}$.

(1) $\mathfrak{F}$ *is a linear subspace over* $\mathbb{Q}$. Obviously $\mathfrak{F}$ is closed under rational multiplication. On the other hand, if $(q \cdot z_n)_{n \in \mathbb{N}}$ and $(q' \cdot z'_n)_{n \in \mathbb{N}}$ are two sequences in $\mathfrak{F}$, then the set $\{ q \cdot z_n + q' \cdot z'_n : n \in \mathbb{N}\}$ is contained in the additive subgroup of $\mathbb{R}$, $q\mathbb{Z} + q'\mathbb{Z}$. Since $q/q' \in \mathbb{Q}$ this subgroup is closed in $\mathbb{R}$ and therefore it must be of the form $p\mathbb{Z}$ for some $p \in \mathbb{Q}$. Thus, $(q \cdot z_n)_{n \in \mathbb{N}} + (q' \cdot z'_n)_{n \in \mathbb{N}}$ belongs to $\mathfrak{F}$.

(2) $\mathfrak{F}$ *is a subring*. This is self–evident.

(3) $\mathfrak{F}$ *is* u–*dense in* $C(\mathbb{N})$. This is an easy consequence of the uniform density theorem contained in [2].

(4) $\mathfrak{F}$ *is not* m–*dense*. It is enough to see that $\mathfrak{F}$ does not satisfy condition (ii) of Proposition 1. Indeed, there is no function $(q \cdot z_n)_{n \in \mathbb{N}}$ in $\mathfrak{F}$ with $0 < q \cdot z_n \leqslant 1/n$ for every $n \in \mathbb{N}$. Otherwise, the sequence of positive numbers $(q \cdot z_n)_{n \in \mathbb{N}}$ contained in the subgroup $q\mathbb{Z}$ would have to converge to 0, but this is impossible because clearly $q\mathbb{Z}$ has no accumulation points.

Finally, we shall show that there is equivalence between u–density and m–density for the subrings of $C(X)$ only in the trivial case.

THEOREM 4. *For a completely regular space* X, *the following conditions are equivalent*:

(a) X *is pseudocompact*.

(b) *Every* u–*dense subring of* $C(X)$ *is* m–*dense*.

Sketch of the proof. Clearly, it is enough to see that (b) implies (a). So, suppose X is not pseudocompact. Then X has a C–embedded copy of $\mathbb{N}$, i.e., a discrete countable subspace of X such that every continuous function on it can be (continuously) extended to X (Gillman–Jerison [4]). We shall denote this copy by $\mathbb{N}$ and take $\mathfrak{F}$ to be the u–dense and not m–dense subring of $C(\mathbb{N})$ constructed in Example 3.

We shall complete the proof when we state that $\mathfrak{F}^{\sim} = \{f \in C(X) : f_{|\mathbb{N}} \in \mathfrak{F}\}$ is a u–dense subring of $C(X)$ that is not m–dense. ∎

Remarks. Note that the same proof is valid if, in the above theorem, instead of subring, we consider one of the following algebraic structures: divisible subring, linear subspace over $\mathbb{Q}$, subgroup, or sublattice. The reason is that the family $\mathfrak{F}$ in Example 3 has each of these properties. Therefore, we can also establish the non–equivalence between u–density and m–density in those cases.

But what is the case for linear subspaces over $\mathbb{R}$ or for subalgebras? Note that here we can not use the same arguments as before since $\mathfrak{F}$ has not any of these structures. We proved in [1], with different techniques, that the analogous result holds for linear subspaces over $\mathbb{R}$. Nevertheless we do not know whether the same is true for the subalgebras of $C(X)$.

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