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Quillen's Theory for Algebraic Models of n–Types

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0. INTRODUCTION

The theory of Quillen model structures is an axiomatic theory with the purpose of understanding what classes of mathematical objects permit a homotopy theory like the one for topological spaces and, the relation among these various homotopy theories as well.

The object of this paper is to describe some new examples in which the axioms for such Quillen model structures are satisfied, and to relate them to the so–called homotopy n–types of spaces.

The starting point is the classical Quillen model structure on the category $\mathbf{Simp}(Gp)$ of simplicial groups. We use it to show that if a category $\underline{C}$ is related to $\mathbf{Simp}(Gp)$ in a suitable way by adjoint functors, then one can obtain a model structure on $\underline{C}$ from the model structure on $\mathbf{Simp}(Gp)$. This result is then applied to obtain a model structure on the category $\mathbf{Simp}^n(Gp)$ of n–simplicial groups, [1], a category which provides algebraic models for all connected homotopy types, and also a model structure on the category $\boldsymbol{n}$–$\mathbf{Hypgd}(Gp)$ of n–hypergroupoids of groups in the sense of Duskin–Glenn, [5]. This category $\boldsymbol{n}$–$\mathbf{Hypgd}(Gp)$ is equivalent to the category $\boldsymbol{n}$–$\mathbf{HXC}(Gp)$ of n–hypercrossed complexes of groups studied by Carrasco and Cegarra in [3] (which provides algebraic models for $(n+1)$–types of spaces) and thus the latter category inherits a model structure as well. In particular, well known low dimensional cases of algebraic models of n–types as crossed modules (1–types) and 2–crossed modules (2–types) inherit, in such a way, a closed model structure.

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1. Lifting Closed Model Structures from **Simp(Gp)**

Let us recall, [6], that a Quillen closed model structure on a category $\underline{C}$ consists of three classes of arrows, weak equivalences, fibrations and cofibrations, such that the following axioms are satisfied:

CM1. $\underline{C}$ has all finite limits and colimits.

CM2. For any pair of composable arrows f and g, if two of the three f, g, gf are weak equivalences, so is the third.

CM3. The classes of weak equivalences, fibrations and cofibrations are closed under retracts.

CM4. (Factorization axiom) Any arrow can be factored as a cofibration followed by a trivial fibration (i.e., an arrow which is a fibration and a weak equivalence), and as a trivial cofibration (i.e., an arrow which is a cofibration and a weak equivalence) followed by a fibration.

CM5. (Lifting axiom) For any commutative diagram in $\underline{C}$

(*)
$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where i is a cofibration, p is a fibration and either i or p is a weak equivalence, the dotted arrow exists (i is said to have the "left lifting property" (LLP) with respect to p, and p is said to have the "right lifting property" (RLP) with respect to i).

Let us consider the category **Simp(Gp)** with its Quillen's closed model structure, [6], where the fibrations are the Kan fibrations, the weak equivalences are those morphisms which induce isomorphisms on the homotopy groups and the cofibrations are defined by the LLP with respect to trivial fibrations, and suppose that $\underline{C}$ is a category which has finite limits and colimits, related to **Simp(Gp)** by an adjunction

(A)
$$\underline{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathbf{Simp}(Gp)$$

with L the left adjoint functor to R.

The aim for this section is to show that, under suitable conditions for this adjoint situation, the category $\underline{C}$ acquires a closed model structure in the

Quillen's sense, which is the "lifted" one from that of **Simp(*Gp*)** in the following sense:

DEFINITION. A morphism f in $\underline{C}$ is said to be a fibration (weak equivalence) if Rf is a fibration (weak equivalence) in **Simp(*Gp*)**. A morphism f in $\underline{C}$ is a cofibration if it has the LLP with respect to the trivial fibrations.

THEOREM. *For the general adjunction* (A) *suppose the functor L preserves weak equivalences, R preserves direct limits and the counit of the adjunction is a natural isomorphism. The category $\underline{C}$ is then a closed model category under the structure proposed by the above definition.*

Sketch of the proof. Axiom CM1 holds by assumption. Axiom CM2 and CM3 (for fibrations and weak equivalences) follow directly from the corresponding facts in **Simp(*Gp*)** using the adjunction. The remaining part for cofibrations follows by a standard formal argument.

To prove the factorization axiom CM4, we use the so-called "small object argument" (see [6]) and the following facts (deduced from the adjoint situation, the characterizations of (trivial) fibrations in **Simp(*Gp*)** and the corresponding universal properties of pushouts and direct limits):

- The functor L preserves cofibrations.
- A morphism f in $\underline{C}$ is a (trivial) fibration if and only if it has the RLP with respect to the family of morphisms $(LF\dot{\Delta}[n] \longrightarrow LF\Delta[n])$ $LF\Delta[n,k] \longrightarrow LF\Delta[n]$ induced by the inclusions $(\dot{\Delta}[n] \hookrightarrow \Delta[n],\ n \geqslant 0)$ $\Delta[n,k] \hookrightarrow \Delta[n]$, $0 \leqslant k \leqslant n$, $n > 0$, with F : **Simp(Sets)** $\longrightarrow$ **Simp(*Gp*)** the free group functor.
- The objects $LF\Delta[n,k]$ and $LF\dot{\Delta}[n]$, $0 \leqslant k \leqslant n$, $n \geqslant 0$, are sequentally small.
- If $A \longrightarrow B$ is a cofibration in $\underline{C}$ and $A \longrightarrow C$ is any morphism, the induced morphism into the pushout $C \longrightarrow B \amalg_A C$ is a cofibration.
- If $C_0 \longrightarrow C_1 \longrightarrow C_2 \longrightarrow \dots$ is a sequence of cofibrations (weak equivalences) in $\underline{C}$, the canonical morphism $C_0 \longrightarrow C_\infty = \lim C_n$ is a cofibration (weak equivalence).
- Given a pushout diagram in $\underline{C}$

$$\begin{tikzcd} LG_\bullet \arrow[r] \arrow[d, "Lf_\bullet"'] & B \arrow[d, "g"] \\ LH_\bullet \arrow[r] & Q \end{tikzcd} ,$$

if $f_\bullet$ is a trivial cofibration in **Simp(*Gp*)**, then g is a trivial cofibration in $\underline{C}$.

Finally, the only non trivial part of the axiom CM5 consists of showing the existence of lifting in commutative diagrams of the form (*) where i is a trivial cofibration anp p is a fibration. The factorization obtained in CM4 (trivial cofibration followed by fibration) and CM2 allow us to obtain the required lifting. ∎

2. Closed Model Structures for Algebraic Model of n–Types

2.1. **n–HXC(Gp)** as a Closed Model Category. The non–abelian version of the classical Dold–Kan's theorem given in [3] allowed to find, by a canonical process of truncation, a new category of algebraic models for $(n+1)$–types. This category, **n–HXC(Gp)**, consists of certain complexes of non–abelian groups, called n–hypercrossed complexes of groups, and it is equivalent to the full subcategory of **Simp(Gp)** formed by those simplicial groups with trivial Moore complex at dimension $> n$, denoted **n–Hypgd(Gp)** since it is just the category of n–hypergroupoids of groups in the sense of Duskin–Glenn, [5].

n–Hypgd(Gp) is a reflexive full subcategory of **Simp(Gp)**, where the reflector functor $P : \mathbf{Simp}(Gp) \longrightarrow n\text{–}\mathbf{Hypgd}(Gp)$, left adjoint to the inclusion functor J, is explicitely given by

$$P(G_\bullet) = \operatorname{cosk}^{n+1}\Big(\frac{G_{n+1}}{H_{n+1}} \rightrightarrows \frac{G_n}{\mathrm{d}_{n+1}(N_{n+1}G_\bullet)} \rightrightarrows G_{n-1} \cdots G_1 \rightrightarrows G_0 \Big),$$

where H_{n+1} is the normal subgroup of G_{n+1} formed by those $x \in G_{n+1}$ such that $\mathrm{d}_i x \in \mathrm{d}_{n-1}(N_{n+1}G_\bullet)$, $0 \leqslant i \leqslant n+1$, with $N_{n+1}G_\bullet$ the Moore complex of $G_\bullet$ at dimension $n+1$.

For this adjoint situation it is clear that $PJ = \mathrm{Id}$, P preserves weak equivalences and J preserves direct limits. Thus, if we define that a morphism f of **n–Hypgd(Gp)** is a fibration (weak equivalence) if Jf is a fibration (weak equivalence) in **Simp(Gp)** and that a morphism is a cofibration if it has the LLP with respect to the trivial fibrations, we have as a direct consequence of the above theorem the following

Theorem. *The category* **n–Hypgd(Gp)** *is a closed model category under the structure above proposed.*

Using now the equivalence of categories between **n–Hypgd(Gp)** and **n–HXC(Gp)** we have

THEOREM. *The category (of algebraic models $(n+1)$-types)* $\boldsymbol{n}$-**HXC**$(\boldsymbol{Gp})$ *is a closed model category.*

Particulary, let us note that 1-**HXC**$(\boldsymbol{Gp})$ is just the category of crossed modules of groups and 2-**HXC**$(\boldsymbol{Gp})$ is that of 2-crossed modules in the sense of Conduché, [4], so that we have

APPLICATION 1. *The category* **XM**$(\boldsymbol{Gp})$ *of crossed modules of groups (2-types) is a closed model category where the fibrations are those morphisms*

$$\Gamma = (f_1, f_0) : (G \xrightarrow{\rho} H) \longrightarrow (G' \xrightarrow{\rho'} H')$$

such that f_1 is surjective and the weak equivalences are those morphisms Γ inducing isomorphisms $\mathrm{Ker}(\rho) \simeq \mathrm{Ker}(\rho')$ *and* $\mathrm{Coker}(\rho) \simeq \mathrm{Coker}(\rho')$.

APPLICATION 2. *The category* **2-XM**$(\boldsymbol{Gp})$ *of 2-crossed modules of groups (3-types) is a closed model category where the fibrations are those morphisms*

$$\Gamma = (f_2, f_1, f_0) : (L \xrightarrow{\varphi} M \xrightarrow{\rho} N) \longrightarrow (L' \xrightarrow{\varphi'} M' \xrightarrow{\rho'} N')$$

such that f_2 and f_1 are surjective and the weak equivalences are those morphisms Γ inducing isomorphisms $\mathrm{Ker}(\varphi) \simeq \mathrm{Ker}(\varphi')$, $\mathrm{Ker}(\rho)/\mathrm{Im}(\varphi) \simeq \mathrm{Ker}(\rho')/\mathrm{Im}(\varphi')$ *and* $\mathrm{Coker}(\rho) \simeq \mathrm{Coker}(\rho')$.

2.2. $\mathbf{Simp}^n(\boldsymbol{Gp})$ AS A CLOSED MODEL CATEGORY. $\mathbf{Simp}^n(\boldsymbol{Gp})$ denotes the category of n-simplicial groups, that is, the category of functors $\boldsymbol{Gp}^{\Delta^{\mathrm{op}} \times \overset{n)}{\cdots} \times \Delta^{\mathrm{op}}}$, so that a n-simplicial group has "n" independent simplicial structures. $\mathbf{Simp}^n(\boldsymbol{Gp})$ is related to **Simp**$(\boldsymbol{Gp})$ (see [1]) by an adjoint situation

$$\mathbf{Simp}^n(\boldsymbol{Gp}) \underset{\overline{W}}{\overset{T}{\leftrightarrows}} \mathbf{Simp}(\boldsymbol{Gp})$$

and then we define that a morphism f in $\mathbf{Simp}^n(\boldsymbol{Gp})$ is a fibration (weak equivalence) if $\overline{W}f$ is a fibration (weak equivalence) of simplicial groups and cofibrations are defined by the LLP with respect to the trivial fibrations.

The counit of the above adjunction is not a natural isomorphism but the proof of the theorem in section 1 works by using the following

LEMMA. *In any pushout diagram in* $\mathbf{Simp}^n(\boldsymbol{Gp})$

$$\begin{tikzcd} \mathcal{T}F\Delta[r,k] \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathcal{T}F\Delta[r] \arrow[r] & P \end{tikzcd} , \qquad 0 \leqslant k \leqslant r,\ r > 0,$$

the morphism f is a trivial cofibration.

This lemma allow us then to assert

THEOREM. *The category* $\mathbf{Simp}^n(\boldsymbol{Gp})$ *is, with the above structure, a closed model category.*

REFERENCES

1. BULLEJOS, M., CEGARRA, A.M. AND DUSKIN, J., On Cat^n-groups and homotopy types, *J. Pure Appl. Algebra* **86** (1993), 135–154.
2. CABELLO, J.G., "Estructuras de Modelos de Quillen para Categorías que Modelan Algebraicamente Tipos de Homotopía de Espacios", Thesis, Univ. de Granada, 1993.
3. CARRASCO, P. AND CEGARRA, A.M., Group theoretic algebraic models for homotopy types, *J. Pure Appl. Algebra* **75** (1991), 195–235.
4. CONDUCHE, D., Modules croisés généralisés de longeur 2, *J. Pure Appl. Algebra* **34** (1984), 155–178.
5. GLENN, P., Realization of cohomology classes in arbitrary exact categories, *J. Pure Appl. Algebra* **25** (1982), 33–107.
6. QUILLEN, D., "Homotopical Algebra", Lect. Notes in Math. 43, Springer-Verlag, Berlin, 1967.