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Fredholm, Riesz and Local Spectral Theory of Multipliers

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INTRODUCTION

A class of bounded linear operators which presents a considerable spectral theory is that of the multipliers of a commutative semi–simple Banach algebra. This class has been introduced in harmonic analysis, for instance in the study of the properties of Fourier tranformations, or in the study of convolution operators and relative problems of characterizations of group algebras. The general theory of multipliers of abstract commutative semi–simple Banach algebras provides a natural abstract context for studying many of the properties of convolution operators acting on the same group algebras. In fact the Gelfand theory permits to represent every multiplier T on a commutative semi–simple Banach algebra A as a bounded continuous complex–valued function $\hat{T}$ defined on a locally compact Hausdorff space $\Delta(A)$, the regular maximal ideal space of A. By means of this representation it appears evident how the spectral properties of T are related to the range of the continuous function $\hat{T}$.

This paper is a survey of recent results on spectral theory of multipliers with a special attention to the various aspects of Fredholm and Riesz theory.

The content is organized as follows:

Section 1 contains some of the basic properties of multipliers of a commutative semi–simple Banach algebra. Some of these results are stated without the assumption of commutativity. This part essentially contains the Wang– Helgason representation theorem of a multiplier as a bounded continuous function and a short discussion on the topologies that we can consider on the maximal ideal space of the multiplier algebra.

In Section 2 we study the concept of relative regularity in the multiplier algebra and we describe the Fredholm theory of multipliers of semi–prime Banach algebras. We also show that for large classes of Banach algebras various kinds of "Fredholm multipliers" coalesce. The most important application of this section is the characterization of Fredholm convolution operators on group algebras.

Section 3 essentially concerns the Riesz theory of multipliers of commutative semi–simple Banach algebras. This theory has interesting connections with the local spectral theory. Also in this section we conclude by giving some applications on group algebras.

1. Basic Properties of Multipliers

Let A be a complex Banach algebra (not necessarily commutative) with or without a unit. A mapping $T: A \longrightarrow A$ is said to be a *multiplier* of A if the equality

$$x(Ty) = (Tx)y$$

holds for all $x, y \in A$. The Banach algebra A is said to be *without order* (or also *proper*) if for all $x \in A$, $xA = \{0\}$ implies $x = 0$, or if for all $x \in A$, $Ax = \{0\}$ implies $x = 0$. Examples of Banach algebras without order are all Banach algebras with an approximate identity and all semi–prime Banach algebras (i.e., Banach algebras without nilpotent nonzero ideals). In particular any semi–simple Banach algebra (i.e., any Banach algebra with radical zero) is without order.

If A is without order, any multiplier T is linear and the equalities

$$T(xy) = x(Ty) = (Tx)y$$

hold for all $x, y \in A$ ([17]), thus the range $T(A)$ of T is a two–sided ideal of A. As a consequence of the closed graph theorem, we also have that T is bounded (see [17], Theorem 1.1.1). Let us denote by $L(A)$ the Banach algebra of all linear bounded operators on A and by $M(A)$ the set of all multipliers of A. In the sequel we shall denote by $\sigma(x, B)$, B any Banach algebra, the *spectrum* of an element $x \in B$.

For a Banach algebra A without order we have ([17]):

I) $M(A)$ is a closed commutative subalgebra of $L(A)$ which contains the identity of $L(A)$.

II) $M(A)$ is an inverse closed subalgebra of $L(A)$, i.e., if T^{-1} does exit,

$T \in M(A)$, then $T^{-1} \in M(A)$.

III) $\sigma(T, L(A)) = \sigma(T, M(A))$, for each $T \in M(A)$.

IV) If A is commutative, then the mapping $x \longmapsto L_x$, L_x the multiplication operator defined by $L_x y = xy$ for all $y \in A$, is a continuous isomorphism of the algebra A onto the ideal $\{L_x : x \in A\}$ in $M(A)$.

V) If A is semi–simple commutative algebra, then also $M(A)$ is semi–simple.

Observe that by identifying A with $\{L_x : x \in A\}$, a commutative Banach algebra A may be thought as an ideal, generally not closed, of $M(A)$.

Given a commutative semi–simple Banach algebra A, let $\Delta(A)$ denote the maximal regular ideal space of A. From Gelfand theory it is well–known that $\Delta(A)$ may be identified with the set of all non trivial multiplicative linear functionals on A. Moreover $\Delta(A)$, equipped with the topology of pointwise convergence (the so called Gelfand topology), is a locally compact Hausdorff space. If we denote by $C_0(\Delta(A))$ the Banach algebra of all continuous complex valued functions defined on $\Delta(A)$ which vanish at infinity, the *Gelfand representation* $x \in A \longmapsto \hat{x} \in C_0(\Delta(A))$, defined by $\hat{x}(m) := m(x)$ for each multiplicative functional $m \in \Delta(A)$, is an isomorphism of A into $C_0(\Delta)$.

Each multiplier of A defines a bounded continuous complex function on the locally compact Hausdorff space $\Delta(A)$. In fact we have

THEOREM 1.1. [23] *Let A be a semi–simple commutative Banach algebra. Then for each $T \in M(A)$ there exists a unique bounded continuous function φ_T on $\Delta(A)$ such that*

$$(T(x))\hat{}\,(m) = \varphi_T(m)\hat{x}(m)$$

holds for all $x \in A$ and all $m \in \Delta(A)$. Moreover $\|\varphi_T\|_\infty \leqslant \|T\|$.

The function φ_T which corresponds to the multiplier T is called the *Wang–Helgason function* of T.

The following theorem gives some information on the spectral properties of multipliers of commutative semi–simple Banach algebras. As usual, the *point spectrum* and the *residual spectrum* of $T \in M(A)$ will be denote by $\sigma_p(T)$ and $\sigma_r(T)$, respectively. Note that $\lambda \in \sigma_r(T)$ if and only if $\lambda I - T$ is injective but has non–dense range.

THEOREM 1.2. [2] *Let A be a commutative semi–simple Banach algebra and $T \in M(A)$. We have:*

i) $\sigma_p(T) \subset \varphi_T(\Delta(A)) \subset \sigma_p(T) \cup \sigma_r(T)$.

ii) *If A has discrete maximal ideal space, then $\sigma_p(T) = \varphi_T(\Delta(A))$.*

Observe that since $M(A)$ is a commutative Banach algebra we can also consider the maximal ideal space $\Delta(M(A))$ of $M(A)$ and the Gelfand transform $\hat{T}$ of the element $T \in M(A)$. There is a relationship between the two functions φ_T and $\hat{T}$: the function φ_T is the restriction of $\hat{T}$ on a certain subset of $\Delta(M(A))$ that may be identified with $\Delta(A)$. In order to explain that let first give some preliminary definitions.

Given an arbitrary commutative Banach algebra B and an ideal J of B, the *hull* of J is defined to be the set

$$h(J) := \{m \in \Delta(B) : m \text{ vanishes on } J\} = \{m \in \Delta(B) : \hat{x}(m) = 0 \text{ for all } x \in J\} .$$

Now, let $m \in \Delta(A)$ and let x be any element of A for which $m(x) \neq 0$. For each $T \in M(A)$ let us define $m^*(T) := m(Tx)/m(x)$. It is easy to check that m^* does not depend of the choice of x and that $m^* \in \Delta(M(A))$. Moreover, m^* is the unique element of $\Delta(M(A))$ whose restriction on A coincides with m. It is evident that given any element $\mu \in \Delta(M(A))$ we have one of the following possibilities: either the restriction $\mu_{|A} = 0$, i.e., $\mu \in h(A)$, the hull of the ideal A in $M(A)$, or the restriction $\mu_{|A} \in \Delta(A)$. The set $\Delta(A)$ may be identified with the set $\Omega := \{\mu \in \Delta(M(A)) : \mu_{|A} \in \Delta(A)\}$. For this reason we shall write

$$\Delta(M(A)) = \Delta(A) \cup h(A) .$$

As we announced before, the restriction of the Gelfand transform $\hat{T}$ on $\Delta(A)$ coincides with the Wang–Helgason function φ_T of T.

The decomposition of $\Delta(M(A))$ above is just a set–theoretic description. Topologically we can say several things: for instance the Gelfand topology of $\Delta(A)$ is the restricton on this set of the Gelfand topology of $\Delta(M(A))$. Moreover $\Delta(A)$ is an open subset of $\Delta(M(A))$ while the hull $h(A)$ is compact [17].

In the maximal regular ideal space of a commutative Banach algebra B one may define another topology weaker than the Gelfand topology, the so called *hull–kernel topology*. First we recall that given a subset $E \subset \Delta(B)$ the *kernel* of E is defined to be the set

$$k(E) := \{x \in B : m(x) = 0 \text{ for all } m \in E\} .$$

The hull–kernel topology is determined by the Kuratowski closure operation: the closure in this topology of $E \subset \Delta(B)$ is the set $\mathrm{cl}(E) := h(k(E))$.

If we consider the two Banach algebras A and $M(A)$, it is easily seen that the hull–kernel topology of $\Delta(A)$ is the relative hull–kernel topology induced by $\Delta(M(A))$. Moreover, with respect to the hull–kernel topology, $\Delta(A)$ is an open subset of $\Delta(M(A))$, whereas this is not always true with respect to the Gelfand topology.

We shall also consider in this paper two important subsets of $M(A)$, the ideal

$$M_0(A) = \{T \in M(A) : \hat{T}_{|\Delta(A)} = \varphi_T \text{ vanishes at infinity on } \Delta(A)\}$$

and the ideal

$$M_{00}(A) = \{T \in M(A) : \hat{T} \text{ vanishes on } h(A)\} .$$

Clearly, we have

$$A \subset M_{00}(A) \subset M_0(A) \subset M(A) ,$$

and all these inclusions may be strict [18].

2. Relative Regularity and Fredholm Theory on $M(A)$

We begin this section by giving some preliminary definitions and basic results on operator theory. Given a linear operator T on a vector space X, the *ascent* of T is defined to be, if it exists, the smallest integer $p = p(T) \geqslant 0$ such that $\mathrm{Ker}\, T^p = \mathrm{Ker}\, T^{p+1}$. If there is no such integer we set $p(T) = \infty$. Analogously the *descent* of T is defined to be, if it exists, the smallest integer $q = q(T) \geqslant 0$ such that $T^q(X) = T^{q+1}(X)$. If there is no such integer we set $q(T) = \infty$. It is well–known [12] that if both the ascent $p(T)$ and the descent $q(T)$ of T are finite, then they are equal. The finiteness of the ascent and the descent of the linear operator T is related to a certain decomposition of the vector space X: if $p = p(T) = q(T) < \infty$, then the decomposition $X = T^p(X) \oplus \mathrm{Ker}\, T^p$ holds and conversely, if for a positive integer n we have $X = T^n(X) \oplus \mathrm{Ker}\, T^n$, then $p(T) = q(T) \leqslant n$ (see [12, Prop. 38.4]).

If X is a Banach space, the condition $0 < p(T) = q(T) = p \in \mathbb{N}$ is equivalent to the condition "$0 \in \mathbb{C}$ is a pole of order p of the vector–valued function $R(\lambda, T) = (\lambda I - T)^{-1}$"; (see [12, Prop. 50.2]).

DEFINITION. Given an algebra $\mathcal{A}(X)$ of endomorphisms on a vector space X, we shall say that $T \in \mathcal{A}(X)$ is *relatively regular* (in $\mathcal{A}(X)$) if there exists an operator $S \in \mathcal{A}(X)$ for which the equalities

$$T = TST\ , \quad STS = S$$

hold. The operator S is also called a *generalized inverse* of T in $\mathcal{A}(X)$.

Observe that in the definition above we may only require $T = TST$. In fact, if $TST = T$, then the operator $S' := STS$ satisfies both the equalities $T = TS'T$ and $S' = S'TS'$. Moreover, if $T = TST$ and $STS = S$, then $P := TS$ and $Q := ST$ are projections with $P(X) = T(X)$ and $\operatorname{Ker} Q = (I - Q)(X) = \operatorname{Ker} T$.

A generalized inverse of an operator, if it exists, is not always uniquely determined. But, as it is easy to show, there exists at most one generalized inverse which commutes with a given $T \in \mathcal{A}(X)$.

In a Banach space X, for a bounded operator $T \in L(X)$, the property of having a commuting generalized inverse $S \in L(X)$ is equivalent to the property $p(T) = q(T) \leqslant 1$ or equivalently to the factorization $T = PU = UP$, where $U \in L(X)$ is invertible and $P \in L(X)$ is a projection [20].

If we consider the special case of a multiplier of a (not necessarily commutative) Banach algebra without order we can say more:

THEOREM 2.1. [20] *If $T \in M(A)$, A a Banach algebra without order, then the following properties are equivalent*:

i) *T has a commuting generalized inverse $S \in L(A)$.*

ii) *T has a generalized inverse $S \in M(A)$.*

iii) *$A = T(A) \oplus \operatorname{Ker} T$, i.e. $p(T) = q(T) \leqslant 1$.*

iv) *$T = PU = UP$ where $U \in M(A)$ is invertible and $P \in M(A)$ is idempotent.*

Now, if A is a semi–prime Banach algebra, $T \in M(A)$ and $x \in \operatorname{Ker} T^2$, we have

$$(Tx)y(Tx) = T(xT(xy)) = T^2(xyx) = (T^2x)yx = 0\ ,$$

for any $y \in A$; hence $Tx = 0$ and $\operatorname{Ker} T^2 \subset \operatorname{Ker} T$. The opposite inclusion is trivially verified for each linear operator on a vector space, so for each $T \in M(A)$ we have $p(T) \leqslant 1$. From that it follows:

THEOREM 2.2. *For a semi–prime Banach algebra A all the statements of the previous theorem are equivalent to the property $q(T) \leqslant 1$, i.e. $T^2(A) = T(A)$.*

Now, let us denote by $\Phi_+(A)$ the class of all *upper semi–Fredholm operators* and by $\Phi_-(A)$ the class of all *lower semi–Fredholm operators*, defined as follows

$$\Phi_+(A) := \{T \in L(A) : \alpha(T) := \dim \operatorname{Ker} T < \infty,\ T(A) \text{ is closed}\}\ ,$$

$$\Phi_-(A) := \{T \in L(A) : \beta(T) := \operatorname{codim} T(A) < \infty,\ T(A) \text{ is closed}\}\ .$$

The class of all *Fredholm operators* on A is defined as $\Phi(A) := \Phi_+(A) \cap \Phi_-(A)$. We recall that, by the so called Atkinson characterization of Fredholm operators, $\Phi(A)$ coincides with the class of all operators of $L(A)$ invertible modulo $K(A)$, the two–sided ideal of all compact operators on A [12]. Finally, let $\Phi_0(A)$ denote the class of all Fredholm operators T on A having index $\operatorname{ind}(T) := \alpha(T) - \beta(T) = 0$.

An elementary property of a multiplier $T \in M(A)$ of a semi–prime Banach algebra is that $T(A) \cap \operatorname{Ker} T = \{0\}$. A direct consequence of this disjointness is

(1) $$\Phi_0(A) \cap M(A) \subset \Phi(A) \cap M(A) = \Phi_-(A) \cap M(A) \subset \Phi_+(A) \cap M(A)\ .$$

The class $\Phi_0(A) \cap M(A)$ may be characterized in a very useful way. Let $K_M(A)$ denote the ideal in $M(A)$ of all multipliers which are compact operators, and let

$$\Phi_M(A) := \{T \in M(A) : T \text{ is invertible in } M(A) \text{ modulo } K_M(A)\}\ .$$

The next theorem shows that we have a sort of "Atkinson characterization" of $\Phi_0(A) \cap M(A)$:

THEOREM 2.3. [6] *Let A be a semi–prime Banach algebra and let $T \in M(A)$. Then the following statements are equivalent:*

i) $T \in \Phi_0(A) \cap M(A)$.

ii) $T \in \Phi_M(A)$.

iii) *T is relatively regular in $M(A)$, and $\alpha(T)$ and $\beta(T)$ are both finite.*

iv) *$T = R + K$, where $R \in M(A)$ is bijective and $K \in M(A)$ is finite dimensional.*

v) *$T = UV$, where $U \in \Phi_M(A)$ is idempotent and $V \in \Phi_M(A)$ is invertible.*

The characterization above has a special interest in the case of a commutative algebra. In fact in this case the Calkin algebra $L(A)/K(A)$ is not commutative; however $M(A)$ and $K_M(A)$ are commutative. Moreover, as we shall see later, in several applications there are concrete models of $M(A)$ and $K_M(A)$.

Note that the property $p(T) \leqslant 1$ has some interesting consequences on the index of a multiplier: if $T \in \Phi(A) \cap M(A)$ we have two possibilities (see [12, Theorem 51.1]):

a) $\operatorname{ind} T = 0$ and $p(T) = q(T) \leqslant 1$.

b) $\operatorname{ind} T < 0$ and $q(T) = \infty$.

In [4] it is shown that if $A = \mathcal{A}(\mathbb{D})$ is the disc algebra and T_g is the multiplication operator by $g(z) = z$, $z \in \mathbb{D}$, then T_g has index $= -1$.

Hence the inclusion $\Phi_0(A) \cap M(A) \subset \Phi(A) \cap M(A)$ may be strict. However, as we shall see later, for a large class of algebras the two sets coincide.

We recall that given a semi-prime algebra $\mathcal{A}$, it is possible to define the *socle* of $\mathcal{A}$ [10] as the sum of all minimal right ideals (it is also equal to the sum of the minimal left ideals, so it is a two-sided ideal in $\mathcal{A}$) or (0) if there are none. Let $\operatorname{Min}(\mathcal{A})$ be the set of all *minimal idempotents* of $\mathcal{A}$, i.e.

$$\operatorname{Min}(\mathcal{A}) := \{e \in \mathcal{A} : 0 \neq e = e^2,\ eAe = \mathbb{C}e\} .$$

The socle can be characterized in terms of the elements of $\operatorname{Min}(\mathcal{A})$:

$$\operatorname{soc} \mathcal{A} = \left\{ \textstyle\sum_{j=1}^{n} Ae_j : n \in \mathbb{N},\ e_j \in \operatorname{Min}(\mathcal{A}) \right\} .$$

In case $\operatorname{Min}(\mathcal{A})$ is empty, we set $\operatorname{soc} \mathcal{A} = (0)$.

Now, let us consider again a semi-prime Banach algebra A and $T \in M(A)$. Note that each minimal idempotent e is an eigenvector of T: in fact, $Te = T(e^3) = e(Te)e \in \mathbb{C}e$. This fact, and the disjointness $T(A) \cap \operatorname{Ker} T = \{0\}$, easily implies that $\operatorname{soc} A \subset T(A) \oplus \operatorname{Ker} T$.

A consequence of the last inclusion is the following theorem.

THEOREM 2.4. [6] *If A is a semi-prime Banach algebra with a dense socle, then*

$$\Phi_M(A) = \Phi_0(A) \cap M(A) = \Phi(A) \cap M(A) = \Phi_-(A) \cap M(A) = \Phi_+(A) \cap M(A) . \tag{2}$$

We shall see later that the last result in the commutative case may be extended to a larger class of Banach algebras. We recall that a commutative Banach algebra A is said to be *Tauberian* if the set $\{x :$ the Gelfand transform $\hat{x}$ has compact support on $\Delta(A)\}$ is norm dense in A. Again, a commutative Banach algebra A is said to be *regular* if for every closed subset $F \subset \Delta(A)$, and every $m_0 \in \Delta(A) \setminus F$ there is an element $x \in A$ for which $m_0(x) \neq 0$ while $m(x) = 0$ for all $m \in F$. Note that each commutative Banach algebra A with a dense socle has discrete maximal ideal [10], so by the Silov idempotent theorem these algebras are regular. Moreover, since the isolated points of $\Delta(A)$ support the minimal idempotents of A, if A has a dense socle then A is also Tauberian.

Next we want to describe shortly the multiplier algebra for the group algebra $A = L_1(G)$, G a locally compact abelian group. First we recall some well–known facts of the Gelfand theory for this algebra. The regular maximal ideal space of the commutative semi–simple Banach algebras $A = L_1(G)$, G a locally compact abelian group, or of $A = L_p(G)$, $1 \leqslant p < \infty$ and G compact, is the set of all continuous characters on G, i.e., the dual group $\hat{G}$ ([13]). Note that the dual group $\hat{G}$ is discrete if G is compact. The Gelfand transform $\hat{f}$ of an element in anyone of these algebras is the so–called *Fourier–transform* of f defined by

$$\hat{f}(\gamma) = \int_G (t^{-1}, \gamma) f(t) \, \mathrm{d}\lambda(t) \ , \qquad (\gamma \in \hat{G}) \ .$$

Moreover, the Gelfand topology on the regular maximal ideal space $\Delta(A) = \hat{G}$ coincides with the usual topology on the dual group $\hat{G}$.

The algebra of multipliers of the group algebra $L_1(G)$, G a locally compact abelian group, may be characterized in a precise way [17]: for each $T \in M(L_1(G))$ there exists an unique measure $\mu \in M(G)$ such that $T = T_\mu$, where $T_\mu : L_1(G) \longrightarrow L_1(G)$ denotes the convolution operator defined by $T_\mu(f) := \mu * f$, $f \in L_1(G)$. Moreover $M(L_1(G))$ is isometrically isomorphic to $M(G)$.

An obvious consequence of the previous theorem and of Theorem 1.4 is that for each multiplier $T = T_\mu$ of $L_1(G)$ the Wang–Helgason function φ_T on $\Delta(A)$ coincides with the Fourier–transform $\hat{\mu}$ on $\hat{G}$.

The following relevant result of abstract harmonic analysis, due to Host and Parreau [14], characterizes the convolution operators on $L_1(G)$ having a closed range.

THEOREM 2.5. *The convolution operator $T_\mu : L_1(G) \longrightarrow L_1(G)$, G a locally compact abelian group, has a closed range if and only if there exists an invertible measure $\nu \in M(G)$ and an idempotent measure τ such that $\mu = \nu * \tau$, i.e., T_μ is the product of an invertible multiplier and an idempotent multiplier.*

By Theorem 2.2, the Host and Parreau's theorem may be phrased as follows:

$$T_\mu \text{ has a closed range} \Leftrightarrow T_\mu^2(A) = T_\mu(A) \ .$$

In a abstract setting the last equivalence may be not true: generally given a multiplier $T \in M(A)$ of a semi-prime Banach algebra, the assumption $T(A)$ closed does not imply $T^2(A) = T(A)$, or equivalently $p(T) = q(T) \leqslant 1$. For instance, let $A = \mathcal{A}(\mathbb{D})$ be the disc algebra and let T_g be the multiplication operator by $g(z) = z$, $z \in \mathbb{D}$. Then $\text{ind}(T_g) = -1$ and hence, as observed above, $q(T_g) = \infty$, while the operator T_g has a closed range.

Observed that, if A is a semi-prime Banach algebra, the implication

$$T^2(A) = T(A) \implies T(A) \text{ is closed}$$

is always true. In fact, in this case we have $p(T) = q(T) \leqslant 1$ and hence, by Theorem 2.1, $A = T(A) \oplus \text{Ker}\, T$ which implies the closedness of $T(A)$ (see [12], Proposition 50.2).

It is reasonable to conjecture that the Host and Parreau's theorem may be extended to more general algebras. For instance, by assuming some structural properties on A, say A regular and Tauberian, then the equivalence

$$T^2(A) = T(A) \iff T(A) \text{ is closed}$$

could be true for each multiplier $T \in M(A)$.

In [5, Theorem 4.2], it is shown that given a semi-simple regular Tauberian Banach algebra A and $T \in M(A)$, then the equivalences

$$(3) \qquad T^2(A) = T(A) \iff T^2(A) \text{ is closed} \iff T(A) \oplus \text{Ker}\, T \text{ is closed}$$

hold. An interesting consequence of these equivalences is obtained when $T \in M(A)$ is injective. In fact if $T(A)$ is closed and T is injective, then $T(A) \oplus \text{Ker}\, T$ is trivially closed, so $T^2(A) = T(A)$ or equivalently, by Theorem 2.2, $A = T(A) \oplus \text{Ker}\, T = T(A)$. Hence T is surjective. Conversely if T is surjective, since $T(A) \cap \text{Ker}\, T = \{0\}$, we have T injective. We have proved:

THEOREM 2.6. *Let A be a regular semi-simple commutative Tauberian Banach algebra and suppose that $T \in M(A)$ has closed range. Then T is surjective if and only if T is injective.*

Another direct consequence of (3) is the extension of (2) to semi-simple commutative regular Tauberian Banach algebras. In fact, for any multiplier $T \in \Phi_+(A) \cap M(A)$ the sum $T(A) \oplus \text{Ker}\, T$ is closed, so $q(T) \leqslant 1$ and hence $A = T(A) \oplus \text{Ker}\, T$. This shows that $T \in \Phi_0(A) \cap M(A)$:

THEOREM 2.7. [5] *Suppose A is a regular semi-simple commutative Tauberian Banach algebra. Then*

$$\Phi_M(A) = \Phi_0(A) \cap M(A) = \Phi(A) \cap M(A) = \Phi_-(A) \cap M(A) = \Phi_+(A) \cap M(A).$$

In [16] Kamowitz has shown that if A is a commutative semi-simple Banach algebra having maximal regular space $\Delta(A)$ with no isolated points, then $T \in M(A)$ is compact if and only if $T = 0$. If A is also regular and Tauberian, by Theorem 2.7, any semi-Fredholm multiplier T belongs to $\Phi_M(A)$. The Kamowizt's result then implies that T is invertible. With an assumption of connectedness we can say more:

THEOREM 2.8. [5] *Suppose A is a regular semi-simple commutative Tauberian Banach algebra with connected maximal idela space $\Delta(A)$ and $0 \neq T \in M(A)$. Then the following statements are equivalent:*

i) *T is semi-Fredholm.*

ii) *T^2 has closed range.*

iii) *T is invertible.*

The most natural application of the theory developed in this section is the complete description of Fredholm theory for convolution operators on the group algebra $L_1(G)$. It is well-known that $L_1(G)$ is a regular Tauberian Banach algebra. Taking into account the Host and Parreau's theorem, we have the following.

THEOREM 2.9. [5] *Let G be a locally compact abelian group and $\mu \in M(G)$ a non-zero regular complex Borel measure on G. For the convolution operator $T_\mu : L_1(G) \longrightarrow L_1(G)$ consider the following statements:*

i) *μ is invertible in $M(G)$.*

ii) *T_μ is semi-Fredholm.*

iii) *T_μ is Fredholm.*

iv) *T_μ is Fredholm of index zero.*

v) *μ is invertible in $M(G)$ modulo the compact multipliers.*

vi) *$\mu * L_1(G)$ is closed and $\mu \neq 0$.*

vii) *$\mu * \mu * L_1(G)$ is closed.*

viii) *μ is the product of a nonzero idempotent measure and an invertible measure.*

Then the following implications hold:

$$\text{i)} \implies \text{ii)} \Leftrightarrow \text{iii)} \Leftrightarrow \text{iv)} \Leftrightarrow \text{v)} \implies \text{vi)} \Leftrightarrow \text{vii)} \Leftrightarrow \text{viii)}$$

If the dual group $\hat{G}$ has no isolated points, then ii) $\Longrightarrow$ i).
If the dual group $\hat{G}$ is connected, then all the statements i) – viii) *are equivalent.*

In [9] it has been shown that if G is a compact abelian group then $K_M(L_1(G)) \simeq L_1(G)$. In this case the socle of $L_1(G)$ is dense in $L_1(G)$ and coincides with the ideal $P(G)$ of all trigonometric polynomials [13]. This ideal is also the socle of the multiplier algebra $M(G)$ [10]. In the next section we shall see that the equality $\operatorname{soc} A = \operatorname{soc} M(A)$ holds for each semi-simple commutative Banach algebra.

Observe that if G is compact, the ideal $F_M(L_1(G))$ of all finite dimensional convolution operators on $L_1(G)$ may be identified with $P(G)$ [13]. Since $K_M(L_1(G)) \simeq L_1(G)$ and $F_M(L_1(G)) \simeq P(G)$ have the same set of projections, the set of all elements in $M(G)$ invertible modulo $L_1(G)$ is the same as the set of all elements in $M(G)$ invertible modulo $P(G)$ (that is also a consequence of the inclusions (5) of the next section).

THEOREM 2.10. *Let G be a compact abelian group and $\mu \in M(G)$ a non-zero regular complex Borel measure on G. For the convolution operator $T_\mu : L_1(G) \longrightarrow L_1(G)$ the statement* v) *of Theorem* 2.9 *may be replaced by*
v′) *μ is invertible in $M(G)$ modulo $L_1(G)$,*
or equivalently
v″) *μ is invertible in $M(G)$ modulo $P(G)$.*
Moreover all the statements ii) – v) *are equivalent to the following statement*:
ix) $\mu = \nu + \varphi$, $\nu \in M(G)$ *invertible and* $\varphi \in L_1(G)$ (*or* $\varphi \in P(G)$).

A class of commutative Banach algebras whose multiplier theory has certain interest is the class of Banach algebras with an orthogonal basis. We recall that a sequence $\{e_k\}$ of elements of a Banach algebra A is said to be an *orthogonal basis* of A if the following two conditions are verified

i) $e_k e_j = \delta_{kj} e_k$ for all $k, j \geqslant 1$.
ii) For each $x \in A$ there exists a unique sequence $\{\lambda_k(x)\}$ of scalars such that

$$x = \lim_{n \to \infty} \sum_{k=1}^{n} \lambda_k(x) e_k = \sum_{k=1}^{\infty} \lambda_k(x) e_k .$$

Trivially any Banach algebra with an orthogonal basis is commutative. Moreover, since the coefficients λ_k depend upon x, it is easy to check that any λ_k is a multiplicative linear functional on A. It is easily seen that any Banach algebra with an orthogonal basis is semi-simple. Moreover, any Banach algebra

A with an orthogonal basis $\{e_k\}$ has a dense socle, because the set of all minimal projections $\mathrm{Min}\,A$ coincides with the sequence $\{e_k\}$. Hence the maximal ideal space $\Delta(A)$ is discrete and, by the Silov idempotent theorem, A is also regular.

In the following we give some examples of Banach algebras with an orthogonal basis.

i) The sequence algebras ℓ_p, where $1 \leqslant p < \infty$, and c_0 (with respect to pointwise operations). An orthogonal basis of these algebras is given by the standard basis $\{u_k\}$, where $u_k := (\delta_{kj})_{j=0,1,\dots}$.

ii) The algebras $L_p(\Gamma)$, $1 < p < \infty$, Γ the circle group, with convolution as multiplication. In fact the sequence $\{e_k\}$, where $e_k(z) := z^k$, $z \in \Gamma$, $k \in \mathbb{Z}$, is an orthogonal basis for $L_p(\Gamma)$ (see [15]).

iii) The so-called *Hardy algebra* $H^p(\mathbb{D})$, $1 < p < \infty$, where $\mathbb{D}$ is the closed unit disc of $\mathbb{C}$. The multiplication on $H^p(\mathbb{D})$ is defined by

$$(f*g)(x) := (2\pi i)^{-1} \int_{|z|=r} f(z)\, g(xz^{-1})\, z^{-1}\, dz$$

where $f, g \in H^p(\mathbb{D})$, $|x| < r < 1$. If we let $e_k(z) := z^k$, $z \in \mathbb{D}$, $k \in \mathbb{N}$, the sequence $\{e_k\}$ is an orthogonal basis for $H^p(\mathbb{D})$ [15].

iv) Any commutative separable H*-algebra (see [17]).

The framework of Banach algebras with an orthogonal basis seems to be the more natural abstract context for unifying the study of multipliers of several algebras.

The next theorem shows that the multipliers of these algebras may be essentially thought as bounded complex sequences:

THEOREM 2.11. [3] *Let A be a Banach algebra with an orthogonal basis $\{e_k\}$. Then there exists a continuous isomorphism Ψ of $M(A)$ onto a subalgebra of ℓ_∞. If the basis $\{e_k\}$ is unconditional, then the map Ψ is a isomorphism of $M(A)$ onto ℓ_∞.*

If the basis $\{e_k\}$ is unconditional, the Fredholm multipliers of A may be characterized in a very simple way:

THEOREM 2.12. [3] *Let A be a Banach algebra with an orthogonal unconditional basis $\{e_k\}$ and let $\{\lambda_k T(e_k)\}$ be the sequence associated with $T \in M(A)$. Then T is a Fredholm multiplier of A if and only if there exists a* *bounded sequence* $\{v_k\}$ *such that* $\lim_{k\to\infty} v_k \cdot \lambda_k(Te_k) = 1$.

3. Riesz Multipliers and Local Spectral Theory

The first part of this section is devoted to some characterizations of Riesz operators defined on a complex infinite dimensional Banach space X This class of operators is defined by assuming as axiom a well–known property of a compact operator: $T \in L(X)$ is said to be a *Riesz operator* if $\lambda I - T \in \Phi(X)$ for all $\lambda \in \mathbb{C} \setminus 0$. Every Riesz operator has a finite or a countable spectrum $\sigma(T)$ which can cluster at most at 0. Moreover each spectral point $\lambda \neq 0$ is an eigenvalue of T [12].

There are several ways of characterizing Riesz operators. For instance, Riesz operators have been characterized by Ruston as those operators for which $\lambda I - T$ is invertible in $L(X)$ modulo the compact operators, for every $\lambda \in \mathbb{C} \setminus 0$. Riesz operators may also be characterized as those operators $T \in L(X)$ for which all non–zero points of the spectrum are poles of finite rank of the resolvent [12].

In order to give other characterizations of this class of operators, we need first to introduce some basic concepts of local spectral theory. If $T \in L(X)$ and $x \in X$, the *local resolvent set* $\rho_T(x)$ is defined to be the union of all open subsets of $\mathbb{C}$ on which the equation $(\lambda I - T)x(\lambda) = x$ admits an analytic solution $x(\lambda)$. The *local spectrum* $\sigma_T(x)$ is defined to be the complement $\mathbb{C} \setminus \rho_T(x)$. Given a closed subset F of $\mathbb{C}$, the *analytic spectral subspace* is defined to be the space $X_T(F) := \{x \in X : \sigma_T(x) \subset F\}$. Finally, an operator $T \in L(X)$ is said to be *decomposable* if for any open covering $\{U_1, U_2\}$ of the complex plane $\mathbb{C}$ there are two T–invariant closed linear subspaces Y_1 and Y_2 of X such that $Y_1 + Y_2 = X$ and $\sigma(T_{|Y_k}) \subset U_k$ for $k = 1, 2$. Note that, since every bounded operator on a Banach space with totally disconnected spectrum is decomposable, any Riesz operator is decomposable.

THEOREM 3.1. *Let X be a Banach space and $T \in L(X)$. Then the following statements are equivalent*:

i) *T is a Riesz operator.*
ii) *$\sigma(T) \setminus 0$ is discrete and the restriction $T_{|Y}$ of T on any T–invariant infinite dimensional closed subspace Y is not bijective.*
iii) *T is decomposable and for all closed subsets $F \subset \mathbb{C} \setminus 0$ the spectral subspaces $X_T(F)$ are finite dimensional.*

The equivalence i) $\Leftrightarrow$ ii) above is quoted in [1]; the equivalence i) $\Leftrightarrow$ iii) may be found in [7].

Next, we want to consider the case of a multiplier T of a commutative semi–simple Banach algebra A. In order to characterize Riesz multipliers we first need to give some information on the Fredholm theory of abstract Banach algebras (the monograph [10] is our main reference). For simplicity, we shall only consider these concepts on commutative Banach algebras.

If $\mathcal{A}$ is a commutative Banach algebra with unit and J is a two–sided ideal of A, it is well–known that

$$x \in A \text{ is invertible modulo } J \Leftrightarrow x \in A \text{ is invertible modulo } k(h(J)) .$$

If $\mathcal{A}$ is semi–simple with socle $\operatorname{soc}\mathcal{A}$, then it is evident that the invertibility in $\mathcal{A}$ modulo $\operatorname{soc}\mathcal{A}$ is equivalent to the invertibility modulo any ideal J for which

$$\operatorname{soc}\mathcal{A} \subset J \subset k(h(\operatorname{soc}\mathcal{A})) . \tag{4}$$

An ideal J for which the inclusion $J \subset k(h(\operatorname{soc}\mathcal{A}))$ holds is said to be an *inessential ideal* of $\mathcal{A}$. The elements of $\mathcal{A}$ invertible modulo an inessential ideal J are called *J–Fredholm elements* of $\mathcal{A}$ and the set of these elements will be denoted by the symbol $\Phi_J(\mathcal{A})$. The inessentiality of an ideal has been characterized by Smyth [22] in terms of the spectrum of its elements: an ideal J is inessential if and only if for each $x \in J$ the spectrum is finite or a sequence which converges to 0. If we take $\mathcal{A} = M(A)$, by (III) of §.1, then the ideal $K_M(A)$ is an inessential ideal, thus the set $\Phi_M(A)$ introduced in §.2 is the set of all $K_M(A)$–Fredholm elements of $M(A)$.

In order to develop a Fredholm theory of the multiplier algebra $M(A)$, we need the following result:

THEOREM 3.2. [7] *Let A be a commutative semi–simple Banach algebra. Then* $\operatorname{soc} A = \operatorname{soc} M(A)$.

In [10] it is shown that a commutative Banach algebra $\mathcal{A}$ has discrete maximal regular ideal if and only if $\mathcal{A} = k(h(\operatorname{soc}\mathcal{A}))$ (a Banach algebra which verifies the last equality is called a *Riesz algebra*). From that it easily follows that any commutative Banach algebra A with a discrete maximal ideal space $\Delta(A)$ is an inessential ideal in $M(A)$. Clearly, in this case we have the following inclusions:

$$\operatorname{soc} A = \operatorname{soc} M(A) \subset A \subset k_{M(A)}(h_{M(A)}(\operatorname{soc} A)) = k_{M(A)}(h_{M(A)}(\operatorname{soc} M(A))) , \tag{5}$$

and

(6) $$\operatorname{soc} A \subset K_M(A) \subset k_{M(A)}(h_{M(A)}(\operatorname{soc} A)) ,$$

(where it is evident that the subscript means that the hull and the kernel are considered with respect to $M(A)$). From the inclusions (5) and (6) it easily follows the equalities

$$\Phi_M(A) = \Phi_{\operatorname{soc} A}(M(A)) = \Phi_A(M(A)) .$$

Note that the equality $\Phi_M(A) = \Phi_{\operatorname{soc} A}(M(A))$ holds without any assumption of discreteness of $\Delta(A)$.

The following theorem lists several characterizations of Riesz multipliers of semi–simple commutative Banach algebras. Note that if T is a Riesz multiplier of a commutative semi–simple Banach algebra, then $T \in M_0(A)$ [7].

THEOREM 3.3. [7] *If A is a semi–simple commutative Banach algebra and $T \in M(A)$, then the following statements are equivalent:*

i) *T is a Riesz operator.*

ii) *The multiplication operator $L_T : S \in M(A) \longrightarrow TS \in M(A)$ is a Riesz operator.*

iii) *$T \in M_0(A)$ and $L_T : S \in M_0(A) \longrightarrow TS \in M_0(A)$ is a Riesz operator.*

iv) *$T \in k_{M(A)}(h_{M(A)}(\operatorname{soc} A))$.*

v) *For every $\lambda \in \mathbb{C} \setminus 0$ we have $\lambda I - T \in \Phi_M(A) = \Phi_{\operatorname{soc} A}(M(A))$.*

vi) *$\sigma(T)$ is finite or a countable set which clusters at 0 and the restriction $T_{|J}$ of T on any T–invariant infinite dimensional closed ideal J is not invertible.*

vii) *T is decomposable and for all closed subsets $F \subset \mathbb{C} \setminus 0$ the spectral subspaces $A_T(F)$ are finite dimensional.*

In order to give other characterizations of Riesz multipliers we need to introduce some other concepts of local spectral theory.

An operator $T \in L(X)$, X a Banach space, is said to have the *weak 2–Spectral Decomposition Property* (shortly *weak 2–SDP*) if, for any open covering $\{U_1, U_2\}$ of the complex plane $\mathbb{C}$ there are two T–invariant closed linear subspaces Y_1 and Y_2 of X such that the sum $Y_1 + Y_2$ is dense in X and $\sigma(T_{|Y_k}) \subset U_k$ for $k = 1, 2$.

A bounded operator $T \in L(X)$, X a Banach space, is said to be *super–decomposable* whenever, for any open covering $\{U_1, U_2\}$ of the complex plane $\mathbb{C}$, there exists some operator $R \in L(X)$ such that $RT = TR$, $\sigma(T_{|R(X)}) \subset U_1$ and $\sigma(T_{|(I-R)(X)}) \subset U_2$.

It is known that any decomposable operator has the weak 2–SDP. Generally the converse does not hold [8]. An example of a decomposable, but not super–decomposable operator can be found in [11]. Later we shall see that for certain classes of multipliers also the converse holds.

In [21] Neumann has shown that for a multiplication operator $L_a : A \longrightarrow A$, $a \in A$, all the kinds of decomposabilities above defined are equivalent to the property that the Gelfand transform $\hat{a} : \Delta(A) \longrightarrow \mathbb{C}$ is continuous with respect to the hull–kernel topology in $\Delta(A)$. In particular, this result holds for any multiplication operator $L_a : A \longrightarrow A$ of a commutative regular semi–simple Banach algebra: in fact in these algebras the Gelfand topology and the hull–kernel topology in $\Delta(A)$ coincide.

The situation becomes more complicate if, instead of a multiplication operator, we look at the problem of decomposability of a multiplier $T \in M(A)$. For a multiplier T the hull–kernel continuity of $\varphi_T = \hat{T}_{|\Delta(A)}$ is only a necessary condition for the decomposability: in fact there are examples of non–decomposable multipliers, even on regular commutative Banach algebras, for which $\hat{T}_{|\Delta(A)}$ is hull–kernel continuous [18].

Actually a characterization of decomposable multipliers is still missing. The situation becomes partially more favourable if we take $T \in M_0(A)$ or even $T \in M_{00}(A)$ (see[18]). Next, we want to give several characterizations of decomposable multipliers in $M_0(A)$ obtained by imposing some additional topological assumptions on $\Delta(A)$.

First, we recall that a locally compact space Λ is said to be *scattered* if each non–empty compact subset of Λ contains an isolated point. Clearly, every discrete space is scattered and every scattered space is totally disconnected.

THEOREM 3.4. [18],[19] *Let A be a commutative semi–simple Banach algebra with $\Delta(A)$ scattered in the Gelfand topology. Then, for each $T \in M(A)$, the following statements are equivalent*:

(a) $T \in M_0(A)$ *and* $\hat{T}$ *is hull–kernel continuous on* $\Delta(M(A))$.

(b) $T \in M_0(A)$ *and* T *is decomposable.*

(c) $T \in M_0(A)$ *and* T *has the weak* 2–*SDP.*

(d) $T \in M_0(A)$ *and* T *is super–decomposable.*

(e) $T \in M_0(A)$ *and* $\sigma(T) = \hat{T}(\Delta(A) \cup \{0\} = \hat{T}(\Delta(A))^-$.

(f) $T \in M_0(A)$ *and* $\sigma(T)$ *is countable.*

(g) $T \in M_{00}(A)$.

We recall that an operator $T \in L(X)$, X a Banach space, is said to be *meromorphic* if all non-zero point of the spectrum are poles of the resolvent $R(\lambda, T) = (\lambda I - T)^{-1}$. A meromorphic multiplier of a semi-simple Banach algebra may be characterized in a very simple way (see [6] or [7]): a multiplier $T \in M(A)$ is meromorphic if and only if the spectrum $\sigma(T)$ is a finite set or a sequence which converges to zero.

Now, let us consider again the case of Banach algebras wich discrete maximal regular ideal space $\Delta(A)$. In this case, by the inclusions (5), we have

$$k_{M(A)}(h_{M(A)}(\mathrm{soc}\, A)) = k_{M(A)}(h_{M(A)}(A)) = M_{00}(A) .$$

This fact and the mentioned characterization of meromorphic multipliers imply the following theorem.

THEOREM 3.5. *Let A be a commutative semi-simple Banach algebra with discrete maximal regular ideal $\Delta(A)$. Then, for each $T \in M(A)$, all the statements* (a) – (g) *of Theorem* 3.4 *are equivalent to the statements* i) – vii) *of Theorem* 3.3. *Moreover these statements are also equivalent to the following property*:

(*) $T \in M_0(A)$ *and* T *is meromorphic.*

We now apply the abstract theory developed in this section to the group algebra $L_1(G)$. For any locally compact abelian group G, let us consider the ideals in $M(G)$

$$M_0(G) := \{\mu \in M(G) : \hat{\mu} \text{ vanishes at infinity on } \hat{G}\} ,$$

and

$$M_{00}(G) := \{\mu \in M(G) : \hat{\mu} \text{ on } \Delta(M(G)) \text{ vanishes outside of } \hat{G}\} .$$

Clearly, if $A = L_1(G)$, we have $M_0(A) = M_0(G)$ and $M_{00}(A) = M_{00}(G)$. Let δ_0 be the identity measure on $M(G)$. By specializing the results of Theorem 3.5 to the algebra $M(G)$, we obtain the following theorem:

THEOREM 3.6. [4],[18] *Let G be a compact abelian group and $\mu \in M_0(G)$ a non-zero regular complex Borel measure on G. For the convolution operator $T_\mu : L_1(G) \longrightarrow L_1(G)$ the following statements are equivalent*:

i) $\hat{\mu}$ *is hull–kernel continuous on* $\Delta(M(A))$.

ii) T_μ *is decomposable.*

iii) T_μ *has the weak 2–SDP.*

iv) T_μ *is super–decomposable.*

v) T_μ *is a Riesz operator.*

vi) $\lambda\delta_0 - \mu$ *is invertible in* $M(G)$ *modulo* $P(G)$ *for every* $\lambda \in \mathbb{C}\setminus 0$.

vii) T_μ *is meromorphic.*

viii) $\sigma(\mu) = \sigma(T_\mu : L_1(G) \longrightarrow L_1(G)) = \hat{\mu}(\hat{G})^-$.

ix) T_μ *has a countable spectrum.*

x) $\mu \in M_{00}(G)$.

Borrowing a concept from harmonic analysis, we shall say that a multiplier $T \in M(A)$ has a *natural spectrum* if $\sigma(T) = \hat{T}(\Delta(A))^-$. By Theorem 3.4, if $\Delta(A)$ is scattered, T has a natural spectrum if and only if T is decomposable. A multiplier $T \in M(A)$ has *natural local spectra* if $\sigma_T(x) = \hat{T}$ (supp $\hat{x}$) for each $x \in A$. This notion is a slightly stronger than the property of having a natural spectrum. It can be shown that if A has scattered maximal regular ideal space then the two notions coincide [11]. So, if $\Delta(A)$ is discrete the property of having a natural spectrum, as well as other concepts of decomposability (see [11],[19]) characterizes Riesz multipliers.

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