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A First-Order Canonical Set of Generalized Jacobi-Type Variables for Hyperbolic Orbital Motion*

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1. INTRODUCTION

The classical Jacobi variables for elliptic orbital motion are traditionally derived for the standard Kepler problem in the negative–energy case. The derivation process is based on the idea of formulating the Hamiltonian of the problem in polar spherical variables and solving it by integrating the corresponding Hamilton–Jacobi partial differential equation, the integration procedure resorting to the separation–of–variables technique. The starting point, approach and purpose adopted in this *Note* are somewhat different from the traditional ones.

The main concern of this research is the construction of an *analytical, closed–form solution* to the dynamical problem of positive–energy two–body motion governed by the quasi–Keplerian type of Hamiltonian function $\mathcal{H}$ proposed by Deprit [3], p. 138, as the simplest radial intermediary of the first order for the *Main Problem* in Satellite Theory. This Hamiltonian is *completely separable* in the *Hill–Whittaker* chart and takes into account the major first–order secular perturbing effects due to the flattening of an oblate spheroid taken as the central body.

To mention a general feature, radial intermediaries constitute simplified models of one–degree–of–freedom (and therefore, integrable) Hamiltonian approximations to the problem of motion of natural and/or artificial orbiters about *oblate spheroidal primaries* and lead to *more general reference orbits*

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than the purely Keplerian one; they share important, far–reaching analytical properties that render the mathematical operations simpler than those required when using other intermediaries. Significance and utility of intermediaries arises from a twofold interest, both as a working tool for the investigation of certain problems of motion in developing perturbation studies (in satellite and/or planetary theories) from simplified models, and the theoretical implications of their construction, which provides a deeper insight into the nature of some fundamental problems of Celestial Mechanics.

The sought solution will be developed in terms of *elementary circular and hyperbolic functions* under a simple form fitting into the usual geometrical and dynamical pattern of the *Keplerian picture* of the hyperbolic two–body motion, provided that appropriate modifications of the hyperbolic orbital elements are incorporated into their defining relations. As for the time, it will be determined with the help of a generalized Kepler–like equation.

As regards this particular sort of orbits, one should not forget that in principle, at least from a theoretical and conceptual point of view, the analytical treatment of hyperbolic–type orbital motion enjoys in the phase space the same importance as that of the bounded one. In addition to this, it is remembered that in practical applications the nature of the orbit can be occasionally changed by perturbing forces acting during a finite interval of time.

Outstanding precedents in this kind of study can be traced in the literature, also in the field of the Theory of Artificial Earth Satellites. For instance, Hori [7], in his analysis of the hyperbolic motion of an artificial satellite under the potential defining the *Main Problem* in that Theory, introduced a suitable variant of the canonical set of Delaunay variables by adapting the construction that, via the Whittaker method, Brouwer and Clemence [1], Chapter XI, §4 and §9, had developed in the context of elliptic motion. In their turn, Cid, Lahulla and Calvo [2] investigated the same hyperbolic J_2 problem by formulating it in Hill–Whittaker polar nodal variables.

With the aim of obtaining the said solution and producing a set of action– and angle–variables behaving as canonical elements of the Jacobi type for the considered problem, the 6–dimensional phase space will be looked on as the stage on which a suitable reducing symplectic transformation will operate. So the canonical system of differential equations of motion derived from $\mathcal{H}$ will be solved by constructing a *complete solution* S to the corresponding *Hamilton–Jacobi* first–order partial differential equation. To this end, in order to apply the *separation–of–variables* technique, advantage will be taken of a set of nontrivial constants of motion that are readily recognized at first sight by mere inspection of the Hamiltonian and suggest the choice of an adequate set of separation constants.

As a result of the treatment of the problem by Hamilton–Jacobi techniques, a *rectification* (see, e.g., Lánczos [8], Chapter VIII, §2, and Scheck [9], §2.37.1) of the Hamiltonian flow of the autonomous system at issue will be obtained after the canonical transformation by means of which the solution is investigated.

To fully achieve the development and contemplate the representation of the solution in Keplerian language, the required intermediate reckoning work can be carried out by adapting certain classical calculations and derivations of canonical elements for the unperturbed, purely Keplerian motion along a hyperbola to the present study, in which a *perturbing potential proportional to* r^{-2} will be allowed for in the analysis. The way of proceeding is based on the idea of modifying a procedure (classically applied to a pure Kepler problem to derive the elliptic Delaunay elements, as done, e. g., in Deprit [3], pp. 115–118, and for quasi-Keplerian systems in the same article, pp. 124–126) to the considered Hamiltonian $\mathcal{H}$.

As a consequence, the proposed solution will absorb the perturbation effects due to the contribution of the J_2 terms of such a potential. From the present approach one achieves a kind of *Keplerian reduction* on the basic Deprit intermediary, and so the conclusion emerges that this research confirms once again the intrinsic Keplerian nature of the Deprit Hamiltonian.

2. The Basic Hamiltonian and the Transformation

As a starting point, the *canonical set of polar nodal variables* $(r, \theta, \nu; p_r, p_\theta, p_\nu)$, constructed by *Whittaker* and *Hill*, is used to coördinatize the 6–dimensional phase space. The meaning of these variables is the following: r denotes the radial distance from the primary's centre of mass to the small moving point; ν represents the argument of longitude of the ascending node in the equatorial plane; θ is the argument of latitude of the orbiter, reckoned from the ascending node. Their conjugate canonical momenta are interpreted as follows: p_r is the radial velocity of the moving mass, p_ν designates the polar component of the angular momentum, and p_θ denotes the modulus of the total angular momentum. Finally, t refers to the physical time, which will act as the independent variable.

Using these variables, the Hamiltonian of the *Main Problem* in Artificial

Satellite Theory reads

$$\mathcal{M} = \mathcal{H}_0 (r , - , - ; p_r , p_\theta , -) + \varepsilon \, \mathcal{M}_1 (r , \theta , - ; p_r , p_\theta , p_\nu)$$

$$= \frac{1}{2} \left[p_r^2 + \frac{p_\theta^2}{r^2} \right] - \frac{\mu}{r} + \varepsilon \, \frac{\mu \, R_e^2}{4 \, r^3} \left\{ (3 \, c^2 - 1) + 3 \, s^2 \cos 2 \theta \right\} ,$$

where the abbreviations $c \equiv \cos I = p_\nu / p_\theta$ and $s \equiv \sin I$ stand for the usual functions of the inclination $I \equiv I (p_\theta , p_\nu)$. After elimination of the parallax (see Deprit [3]), this Hamiltonian becomes the function

$$\mathcal{H} \equiv \mathcal{H} (r , - , - ; p_r , p_\theta , p_\nu ; \varepsilon) = \mathcal{H}_0 + \varepsilon \, \frac{\mu^2 \, R_e^2}{4 \, r^2 \, p_\theta^2} \, (3 c^2 - 1) ,$$

named as the *Deprit radial intermediary.* Here the Hamiltonian $\mathcal{H}_0$ pertains to a conventional Kepler problem, R_e refers to the mean equatorial radius of the central body, and the (small) dimensionless coefficient $\varepsilon = -J_2$ accounts for the *oblateness* in the gravity field of the primary. A dash has been used in place of a variable to emphasize its explicit absence from $\mathcal{H}$.

As will be seen, the *validity* of the subsequent discussion is not affected by the specific functional form under which the momenta p_θ and p_ν and the perturbation parameter ε occur in the expression of $\mathcal{H}$. Indeed, in the light of the functional dependence of this Hamiltonian, a glance at $\mathcal{H} \equiv \mathcal{H} (r , - , - ; p_r , p_\theta , p_\nu ; \varepsilon)$ shows that the angular coordinates θ and ν are *cyclic* for the intermediary problsl at hand, and so their canonically conjugate momenta are *invariant quantities* throughout the motion:

$$p_\theta = \Theta_0 = \text{const.} , \qquad p_\nu = N_0 = \text{const.}$$

By virtue of the *conservative* nature of the problem (since $\mathcal{H}$ is not explicitly dependent on the time t), the system admits the *first integral of the energy*:

$$\mathcal{H} = K_0 ,$$

K_0 denoting the constant value of the total energy of the Hamiltonian. For definiteness, K_0 will from now on be supposed *positive*, according to the assumption on the hyperbolic nature of the orbit.

Now, the integration of the differential equations of motion derived from $\mathcal{H}$ *amounts to* finding a canonical transformation to a set of constant momenta and all but one constant coordinates, the remaining coordinate being a linear function of the independent variable t. This transformation will be accomplished by means of a suitable scalar *generating function* S; in order to determine S, it will suffice to know a *complete solution* to the *Hamilton–Jacobi* first–order partial differential equation associated with $\mathcal{H}$, namely:

$$\mathcal{H}\left(r\,,\,-\,,\,-\,;\,\frac{\partial S}{\partial r}\,,\,\frac{\partial S}{\partial \theta}\,,\,\frac{\partial S}{\partial \nu}\,;\,\varepsilon\right) \;=\; K_0\,,$$

and the unknown function S generates a symplectic mapping from the phase space of the polar nodal variables,

$$(\,r\,,\,\theta\,,\,\nu\,;\,p_r\,,\,p_\theta\,,\,p_\nu\,) \quad \xrightarrow{\;S\;} \quad (\,Q_K\,,\,Q_\Theta\,,\,Q_N\,;\,K_0\,,\,\Theta_0\,,\,N_0\,)\,,$$

the constant values of the above conserved quantities of the problem (namely: K_0, Θ_0 and N_0) having been chosen as the canonical momenta of the new set of phase variables.

Since any cyclic coordinate is separable and for this Hamiltonian all but one of the coordinates are cyclic, its corresponding Hamilton–Jacobi equation turns out to be *completely separable* in the coordinates chosen, in such a way that its integration can be more easily effected if one seeks a trial solution for S by *separation of variables*, say in the form of a sum of separate functions of the separate coordinates, each function involving just one of the coordinates and one or some of the selected constants:

$$S \;\equiv\; S\,(\,r\,,\,\theta\,,\,\nu\,;\,K_0\,,\,\Theta_0\,,\,N_0\,) \;=\; \theta\,\Theta_0 \;+\; \nu\,N_0 \;+\; W\,(r)\,.$$

Next, upon substituting this trial solution into the equation, the resulting reduced Hamilton–Jacobi equation is

$$\frac{1}{2}\left(\frac{d\,W}{d\,r}\right)^2 \;+\; \frac{\Theta_0^2}{2\,r^2} \;-\; \frac{\mu}{r} \;+\; \varepsilon\;\frac{\mu^2\,R_e^2}{4\,r^2\,\Theta_0^2}\;\left[\,3\;\frac{N_0^2}{\Theta_0^2}\;-\;1\,\right] \;=\; K_0\,.$$

Correspondingly,

$$\left(\frac{d\,W}{d\,r}\right)^2 \;=\; Q\,(\,r\,;\,K_0\,,\,\Theta_0\,,\,N_0\,;\,\varepsilon\,)\,,$$

where

$$Q \equiv Q(r;K_0,\Theta_0,N_0;\varepsilon) = 2K_0 + \frac{2\mu}{r} - \frac{1}{r^2}\left\{\Theta_0^2 + \varepsilon\frac{\mu^2 R_e^2}{2\Theta_0^2}\left[3\frac{N_0^2}{\Theta_0^2} - 1\right]\right\}.$$

Thus the *generating function* (that depends on a combination of variables that mixes the two sets together) can be written compactly in the form

$$S \;\equiv\; S(\,r\,,\,\theta\,,\,\nu\,;\,K_0\,,\,\Theta_0\,,\,N_0\,) \;=\; \theta\,\Theta_0 \;+\; \nu\,N_0 \;+\; \int_{r_0}^{r}\sqrt{Q}\;\;dr\,,$$

the lower limit of integration r_0 being the only (positive) real root of the r–equation given by $Q(r\ ;\ K_0, \Theta_0, N_0\ ;\ \varepsilon) = 0$, which will formalize a condition for $\dot{r} = 0$, i. e., for r to have an extremum.

The implicit equations of the symplectic change of phase variables derived from a complete solution S of the Hamilton–Jacobi equation linked to $\mathcal{H}$ are expressed by the following relations:

$$p_r = \sqrt{Q}\,, \qquad Q_K = \sqrt{\frac{a^3}{\mu}}\ (e \sinh F - F)\,,$$
$$p_\theta = \Theta_0\,, \qquad Q_\Theta = \theta - \Delta_{(\theta)}\, f\,,$$
$$p_\nu = N_0\,, \qquad Q_N = \nu - \Delta_{(\nu)}\, f\,.$$

The quantities and functions involved in the above equations admit a simple interpretation that parallels the usual *Keplerian language*, the formulae bearing a close resemblance to those holding for the standard Kepler problem (specially when the perturbation terms are neglected). With this aim in view one introduces a set of appropriate *subsidiary quantities* $a \equiv a(K_0)$, $e \equiv e(K_0, \Theta_0, N_0\ ;\ \varepsilon)$, $p \equiv p(\Theta_0, N_0\ ;\ \varepsilon)$, $\kappa \equiv \kappa(\Theta_0, N_0\ ;\ \varepsilon)$, $n \equiv n(K_0)$, $\Delta_{(\theta)}(\Theta_0, N_0\ ;\ \varepsilon)$, $\Delta_{(\nu)}(\Theta_0, N_0\ ;\ \varepsilon)$ and the *auxiliary variables* $F \equiv F(r\ ;\ K_0, \Theta_0, N_0\ ;\ \varepsilon)$ and $f \equiv f(r\ ;\ K_0, \Theta_0, N_0\ ;\ \varepsilon)$ by means of the set of relations

$$a = \frac{\mu}{2K_0}\,, \qquad \kappa^2 \equiv \Theta_0^2 + \varepsilon\,\frac{\mu^2 R_e^2}{2\,\Theta_0^2}\left(3\,\frac{N_0^2}{\Theta_0^2} - 1\right) = \mu\, a\,(e^2 - 1)\,,$$

$$e^2 = 1 + \frac{2K_0\,\kappa^2}{\mu^2}\,, \qquad p = a(e^2 - 1) = \frac{\kappa^2}{\mu}\,, \qquad \mu = n^2 a^3\,,$$

$$\Delta_{(\theta)} \equiv \frac{\partial\kappa}{\partial\Theta_0} = \frac{1}{\kappa}\left[\Theta_0 + \varepsilon\frac{\mu^2 R_e^2}{2\Theta_0^3}\left(1 - 6\frac{N_0^2}{\Theta_0^2}\right)\right], \quad \Delta_{(\nu)} \equiv \frac{\partial\kappa}{\partial N_0} = \frac{1}{\kappa}\left[\varepsilon\frac{3\mu^2 R_e^2 N_0}{2\Theta_0^4}\right],$$

$$\frac{a + r}{a\,e} = \cosh F\,, \qquad r = a(e\cosh F - 1)\,, \qquad r = \frac{p}{1 + e\cos f}\,.$$

By virtue of the positivity of the total energy, the preceding eccentricity–like function $e(K_0, \Theta_0, N_0\ ;\ \varepsilon)$ is such that $e > 1$, which leads to a straightforward determination of the root of the r–equation $Q(r\ ;\ K_0, \Theta_0, N_0\ ;\ \varepsilon) = 0$, say: $0 < r_0(K_0, \Theta_0, N_0\ ;\ \varepsilon) = a(e - 1)$. So, keeping in mind the meaning of the zero of Q, one concludes that it is the *perturbed pericentre radial distance*.

These formulae are similar to those holding for a hypothetic Keplerian motion characterized by the above *hyperbolic elements* (a, e, p) with κ as the modified angular momentum magnitude and n as a kind of hyperbolic mean motion.

Notice that the equation for Q_K is one of the type of a *Kepler equation*, which will later serve as the fundamental relation connecting the position *on* the orbit and the time. In addition to this, observe that, by constrast to the classical Jacobi transformation, this equation involves not only r, K_0 and Θ_0 but also the momentum N_0.

On the other hand, although p_r has the same functional form as in the conventional Keplerian case, one should not overlook the fact that it actually depends on N_0 through F or f. Thus, for instance, one will get that

$$p_r^2 + \frac{p_\theta^2}{r^2} = 2\,K_0 + \frac{2\,\mu}{r} - \frac{1}{r^2}\,[\kappa^2 - \Theta_0^2]\,.$$

In the next section the canonical transformation here obtained will be applied to the Deprit intermediary. In so doing, it will be seen that the functional dependence of $\mathcal{H}$ is substantially simplified when formulated in the new Jacobi–like variables.

3. Solution to the Intermediary

To complete the picture, the Deprit Hamiltonian will be reduced by the transformation defined in the preceding section. In the new chart it becomes the function

$$\mathcal{H} \longrightarrow \widetilde{\mathcal{H}} \equiv \widetilde{\mathcal{H}}(-,-,-\,;\,K_0\,,-,-) = K_0 = \frac{\mu}{2\,a}$$

$$\Longrightarrow \quad \frac{d\,Q_K}{d\,t} = \frac{\partial\widetilde{\mathcal{H}}}{\partial K_0} = 1 \quad \Longrightarrow \quad Q_K = t + \text{const.} = t - t_0\,.$$

Thus the transformation performs a canonical reduction of $\mathcal{H}$ to the Hamiltonian $\widetilde{\mathcal{H}}$ corresponding to a hypothetic unperturbed hyperbolic Kepler problem, and the proposed variables absorb all secular variations included in the potential of $\mathcal{H}$.

The integration constant t_0 is an epoch constant depending merely on the instant from which t is measured in the time scale. The choice of t_0 as the instant at which the moving mass performs the *pericentre passage* is a customary practice in Orbital Mechanics studies.

In the *canonical solution* to the equations of motion generated by the reduced Hamiltonian $\widetilde{\mathcal{H}}$ *in the new system* of variables, the only variable that is not a constant of the motion is Q_K, which is equal to the time plus a constant.

It should be emphasized that, as usual in Hamilton–Jacobi theory (see, e. g., Lánczos [8], Chapter VIII, §2; Scheck [9], §2.37.1), the following interpretation can be ascribed to the above results: in the neighbourhood of every point of phase space that is not an equilibrium position, the transformation *smooths the flow* of the autonomous Hamiltonian system derived from $\mathcal{H}$ to a *uniform, rectilinear flow*, producing a bundle of parallel straight lines inclined at an angle of 45° to the time axis.

As in the previous *Note* [4], from the transformation formulae, by solving for the original polar nodal variables, the sought *hyperbolic Keplerian-like solution* to the Deprit intermediary $\mathcal{H}$ can be set up in a *parametric* representation:

Generalized Kepler Equation: $Q_K = \dfrac{1}{n}\,(e\sinh F - F) = t - t_0$,

Radial distance: $r = a\,(e\cosh F - 1) = \dfrac{p}{1 + e\cos f}$,

Radial velocity: $p_r = \sqrt{Q} = \sqrt{\dfrac{\mu}{a}}\,\dfrac{e\sinh F}{(e\cosh F - 1)} = \sqrt{\dfrac{\mu}{p}}\,e\sin f$,

Argument–of–latitude equation: $\theta = Q_\Theta + \Delta_{(\theta)}\,f$,

Magnitude of the angular momentum vector: $p_\theta = \Theta_0$,

Node equation: $\nu = Q_N + \Delta_{(\nu)}\,f$,

Polar component of the angular momentum vector : $p_\nu = N_0$.

Consequently, collecting the preceding details, the resulting variables for the fictitious "hyperbola–like" motion are

$$Q_K = \frac{1}{n}\,(e\sinh F - F) = t - t_0\,, \qquad K_0 = \frac{\mu}{2\,a} = \text{const.},$$
$$Q_\Theta = \theta - \Delta_{(\theta)}\,f = \text{const.}, \qquad \Theta_0 = p_\theta = \text{const.},$$
$$Q_N = \nu - \Delta_{(\nu)}\,f = \text{const.}, \qquad N_0 = \Theta_0\cos I = p_\nu = \text{const.},$$

and, at first sight, $(a, e, I, Q_\Theta, Q_N, t_0, n)$ resemble and bring to mind the standard Keplerian orbital elements of hyperbolic motion.

4. Final Remarks

(1) As in the case of the generalized Delaunay elements considered in the previous papers [4] and [5], the application of these Jacobi–like variables to the hyperbolic Main Problem in Artificial Satellite Theory performs a *removal of the non–trigonometric terms of the first order* occurring in the J_2 part of this Hamiltonian. Comments like those in [4] are also in order now.

(2) The transition from the preceding generalized Jacobi set to the just mentioned Delaunay–type elements,

$$(Q_K, Q_\Theta, Q_N ; K_0, \Theta_0, N_0) \xrightarrow{\hat{S}} (l, g, h ; L, G, H),$$

can be readily accomplished via a simple canonical transformation derived with the help of the generating function

$$\hat{S} \equiv \hat{S}(K_0, \Theta_0, N_0 ; l, g, h) = \frac{\mu}{\sqrt{2 K_0}} l - \Theta_0 g - N_0 h,$$

which is inspired by a proper modification of one considered, e. g., by Garfinkel [6], pp. 64–65, taking into account certain changes of sign due to the hyperbolic nature of the motion.

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