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On the Effect of Erroneous Models on Systems in Presence of Correlated Disturbances and Uncertain Observations

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When a system model is used as an approximation to a physical situation, a series of parameters must be specified. In particular, for estimation problems in linear systems with perturbations, some parameters of the distributions of the noises must be known. In many practical situations, such parameters are not generally known and they must be approximated. Even such an approximation is, sometimes, intentional, for example if it is desirable to simplify the mechanization of the estimates. Since an inexact model will degrade the estimator performance, it is important to evaluate the effect on performance of the approximation made.

The effect of erroneous models on the Kalman filter response has been studied by several authors; for example, Nishimura [4] has considered the effect of an incorrect initial covariance matrix and Heffes [1] has considered errors in the covariance matrices of the initial state and noises.

Some extensions of the Kalman filter have been studied for systems with uncertain observations; Nahi [3] derived the least mean–squared error (LMSE) linear filter for this kind of systems by assuming that the sequence characterizing interruptions in the observations was a scalar binary–valued white noise. Hermoso and Linares [2] have generalized the Nahi's work in cases where the state and measurement noises are correlated. This paper concerns itself with the effect of incorrect parameters on the estimator response in these cases. The kinds of errors considered are those in the initial state and noises covariance matrices of the model.

SYSTEM MODEL

The considered linear discrete-time system has the following description

$$
\begin{aligned}
x(i) &= A(i-1)x(i-1) + W_1(i-1), \quad i > i_0 \\
x(i_0) &= x_0 \\
y(i) &= U(i)C(i)x(i) + W_2(i), \quad i \geq i_0
\end{aligned} \tag{1}
$$

where $x(i)$ is the $n \times 1$ state vector and $y(i)$ is the $p \times 1$ observation vector; $A(i)$ and $C(i)$ are known matrices of appropriate dimensions; $\{W_1(i)\}$ and $\{W_2(i)\}$ are white noise sequences with zero mean and

$$
\begin{aligned}
E[W_1(i)W_1'(l)] &= V_1(i)\delta_{il}, \\
E[W_2(i)W_2'(l)] &= V_2(i)\delta_{il}, \\
E[W_1(i-1)W_2'(l) &= V_{12}(i)\delta_{il};
\end{aligned}
$$

the initial state x_0 is a random vector with mean $\bar{x}_0$ and covariance Q_0; $\{U(i)\}$ is a sequence of independent random variables with $Pr(U(i) = 1) = p(i)$ and $Pr(U(i) = 0) = 1 - p(i)$. We assume that $\{U(i)\}$ is independent of $(x_0, \{W_1(i)\}, \{W_2(i)\})$ and x_0 is independent of $\{W_1(i)\}$ and $\{W_2(i)\}$. Also we assume that $Cov[U(i)C(i)W_1(i-1) + W_2(i)]$, $i > i_0$ and $V_2(i_0)$ are positive definite.

LMSE LINEAR RECURSIVE ESTIMATOR

The recursive formulas for the LMSE linear estimator $\hat{x}(i|i)$ of $x(i)$ given from the observations $y(i_0), \ldots, y(i)$, are derived in Hermoso and Linares [2]:

$$
\begin{aligned}
\hat{x}(i|i) &= \hat{x}(i|i-1) + F(i,i)[y(i) - p(i)C(i)\hat{x}(i|i-1)], \quad i \geq i_0 \\
\hat{x}(i|i-1) &= A(i-1)\hat{x}(i-1|i-1), \quad i > i_0 \\
\hat{x}(i_0|i_0-1) &= \bar{x}_0
\end{aligned} \tag{2}
$$

where $\hat{x}(i_0|i_0-1)$ denotes the estimator of x_0 when a priori observation is not available.

The gain matrix $F(i,i)$ is given by

$$
\begin{aligned}
F(i,i) = {} & [p(i)Q(i|i-1)C'(i) + V_{12}(i)][p(i)(1-p(i))C(i)D(i)C'(i) + \\
& p^2(i)C(i)Q(i|i-1)C'(i) + p(i)C(i)V_{12}(i) + \\
& p(i)V_{12}'(i)C'(i) + V_2(i)]^{-1}
\end{aligned} \tag{3}
$$

being $D(i) = E[x(i)x'(i)]$ and the error covariance matrices are

$$(4)\quad \begin{aligned} &Q(i|i) = [I - p(i)F(i,i)C(i)]Q(i|i-1) - F(i,i)V'_{12}(i), \quad i \geq i_0 \\ &Q(i|i-1) = A(i-1)Q(i-1|i-1)A'(i-1) + V_1(i-1), \quad i > i_0, \\ &Q(i_0|i_0-1) = Q_0 \end{aligned}$$

Effect of Incorrect Covariance Matrices

We study the estimator performance obtained when the above design is based upon incorrect covariance matrices of the initial state and additive noises. A measure of the estimator performance is provided by the actual estimation error covariance matrix.

In the next, we denote $\hat{Q}_0$, $\hat{V}_1(i)$ and $\hat{V}_2(i)$ the covariance matrices of the initial state and noises used in the estimator design rather than the correct Q_0, $V_1(i)$ and $V_2(i)$ respectively. We represent by Q_c and F_c the covariance and gain matrices respectively, computed by sustituting $\hat{Q}_0$, $\hat{V}_1(i)$ and $\hat{V}_2(i)$ in the expressions (3) and (4). We denote by $\hat{x}_c$ the computed estimators when they are derived from (2) by using F_c. Then the actual estimation errors, $e(i|i) = x(i) - \hat{x}_c(i|i)$ and $e(i|i-1) = x(i) - \hat{x}_c(i|i-1)$, are given by

$$\begin{aligned} e(i|i) = [I - p(i)F_c(i,i)C(i)]e(i|i-1) - F_c(i,i)W_2(i) - \\ [U(i) - p(i)]F_c(i,i)C(i)x(i), \quad i \geq i_0 \\ e(i|i-1) = A(i-1)e(i-1|i-1) + W_1(i-1), \quad i > i_0 \\ e(i_0|i_0-1) = x_0 - \bar{x}_0. \end{aligned}$$

The actual covariance matrices of these errors indicate the statistical quality of the computed estimators; using the hypotheses of the model, we have

$$\begin{aligned} Q_a(i|i) = [I - p(i)F_c(i,i)C(i)]Q_a(i|i-1)[I - p(i)F_c(i,i)C(i)]' - \\ [I - p(i)F_c(i,i)C(i)]V_{12}(i)F'_c(i) - F_c(i,i)V'_{12}(i)[I - p(i)F_c(i,i)C(i)]' \\ + F_c(i,i)[p(i)(1-p(i))C(i)D(i)C'(i) + V_2(i)]F'_c(i,i), \quad i \geq i_0 \\ Q_a(i|i-1) = A(i-1)Q_a(i-1|i-1)A'(i-1) + V_1(i-1), \quad i > i_0 \\ Q_a(i_0|i_0-1) = Q_0. \end{aligned}$$

For comparing the actual, Q_a, and computed, Q_c, covariance matrices, we define the functions $E(i|i) = Q_c(i|i) - Q_a(i|i)$ and $E(i|i-1) = Q_c(i|i-1) - Q_a(i|i-1)$. The expressions for Q_c and Q_a yield to the following recurrence relationships for these functions

$$(5)\quad E(i|i) = [I - p(i)F_c(i,i)C(i)]E(i|i-1)[I - p(i)F_c(i,i)C(i)]' + F_c(i,i)\left[\hat{H}(i) - H(i)\right]F_c'(i,i), \quad i \geq i_0$$

$$(6)\quad \begin{aligned} E(i|i-1) &= A(i-1)E(i-1|i-1)A'(i-1) + \hat{V}_1(i-1) - V_1(i-1), \quad i > i_0 \\ E(i_0|i_0-1) &= \hat{Q}_0 - Q_0 \end{aligned}$$

where

$$\begin{aligned} H(i) &= p(i)(1-p(i))C(i)D(i)C'(i) + V_2(i), \\ \hat{H}(i) &= p(i)(1-p(i))C(i)\hat{D}(i)C'(i) + \hat{V}_2(i), \\ \hat{D}(i) &= A(i-1)\hat{D}(i-1)A'(i-1) + \hat{V}_1(i-1), \\ \hat{D}(i_0) &= \hat{Q}_0 + \bar{x}_0\bar{x}_0'. \end{aligned}$$

THEOREM. *If* $Q_0 \leq \hat{Q}_0$, $V_1(i) \leq \hat{V}_1(i)$ *and* $V_2(i) \leq \hat{V}_2(i)$ *for all* i, *then* $Q_a(i|i) \leq Q_c(i|i)$ *and* $Q_a(i|i-1) \leq Q_c(i|i-1)$ *for all* i.

Proof. It is clear that $\hat{H}(i) \geq H(i)$ for all i and hence the second term in the right side of (5) is non-negative definite. If $E(i|i-1) \geq 0$, it is easy to see that the first term on the right side of (5) is non-negative definite and hence it is evident that $E(i|i) \geq 0$. From (6), if $E(i|i) \geq 0$, it is clear that $E(i+1|i) \geq 0$; by induction, since $E(i_0|i_0-1) \geq 0$, we have the desired results. ∎

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