EXTRACTA MATHEMATICAE Vol. 10, Núm. 1, 64–67 (1995)

On the Chebyshev Alternation Theorem †

Manuel Fernández and María L. Soriano

Dpto. de Matemáticas, Univ. de Extremadura, 06071-*Badajoz, Spain*

(Presented by C. Benítez)

AMS *Subject Class.* (1991): 41A65, 46B20 *Received June* 9, 1995

1. Introduction

The Hahn–Banach and Krein–Milman theorems permit us to obtain characterization results of best approximations. Although numerous authors have been used this approach (see e.g. [4, 6, 9]), probably the Singer's book [11] is the main effort directed to the application of functional methods to the approximation theory.

In this paper, we derive a general characterization result of best simultaneous approximations from the Deutsch-Maserick theorem [4]. As an application to the space of continuous functions endowed with the uniform norm, we prove an alternation theorem which generalizes the ones stated by Rémes-Golomb [10, 7] and Ling-McLaughlin-Smith [8].

Let E be a normed space. As usual, B_E (resp. S_E) denotes the closed unit ball (resp. unit sphere) of E with center at the origin. If $A \subset E$, $\operatorname{ext}(A)$ is the set of extremal points of A, and $\operatorname{Re}\lambda$ denotes the real part of the scalar λ.

If L is a subset of E and $x \in E$, the set of best approximations to x from L is

$$P_L(x) := \{u_0 \in L \;:\; \|x - u_0\| = \inf_{u \in L} \|x - u\|\}.$$

To define a criterion for simultaneous approximation, endow $E \times E$ with a norm. Then $u_0 \in L$ is a *best simultaneous approximation* to $x, y \in E$ from L if $(u_0, u_0) \in P_{d(L\times L)}(x,y)$, where $d(L \times L) = \{(u,u) \in E \times E \;:\; u \in L\}$ is the diagonal set of $L \times L$.

We will focus our attention on M-norms on $E \times E$, i.e. those norms defined as $\|(x,y)\| = |(\|x\|, \|y\|)|$, where $|\cdot|$ is a norm in $\mathbb{R}^2$ such that $|r_k| \le |s_k|$,

† A full version will appear in *Atti Sem. Mat. Fis. Univ. Modena*

$k = 1, 2$, implies $|(r_1, r_2)| \le |(s_1, s_2)|$. The most typical example of M-norms is the family of p-norms,

$$\|(x, y)\|_p = (\|x\|^p + \|y\|^p)^{1/p},\ p \ge 1, \quad \|(x, y)\|_\infty = \max(\|x\|, \|y\|).$$

From now on, B' will denote the unit ball of $|\cdot|'$, the dual norm of $|\cdot|$. Finally if $(x, y) \ne (0, 0)$, we set

$$\Delta_{x,y} = \{(r_1, r_2) \in \mathbb{R}^2 \ :\ |(r_1, r_2)|' = 1,\ r_1\|x\| + r_2\|y\| = \|(x, y)\|\}.$$

2. Characterization of Best Simultaneous Approximations

The general characterization theorem follows from [3, Theorem 1] and [4, Theorem 2.5].

THEOREM 1. *Let $E \times E$ be an M-normed space, L a convex set of E, and x, y distinct points of E. Then $u_0 \in L$ is a best simultaneous approximation to x, y from L if and only if there exists $(\varphi, \psi) \in E' \times E'$ such that*

$$(\|\varphi\|, \|\psi\|) \in \Delta_{x-u_0, y-u_0},$$
$$\varphi(x - u_0) = \|\varphi\|\,\|x - u_0\|, \quad \psi(y - u_0) = \|\psi\|\,\|y - u_0\|,$$
$$\mathrm{Re}((\varphi + \psi)(u - u_0)) \le 0, \quad \text{for every } u \in L.$$

In the following, $E = C[a, b]$ will denote the space of real continuous functions on the compact interval $[a, b]$ of $\mathbb{R}$ endowed with the uniform norm. A subset $\{u_1, \ldots, u_n\}$ of E is said to have the *Haar property* if, for every choice of n different points $t_1, \ldots, t_n \in [a, b]$, we have $\det u_j(t_i) \ne 0$, $i, j = 1, \ldots, n$. The n-dimensional linear subspace L, generated by the functions $\{u_1, \ldots, u_n\}$, is called a Haar subspace.

Let $E \times E$ be endowed with an M-norm; $x, y \in L$; L be an n-dimensional Haar subspace of E; and $u_0 \in L$. We will say that (u_0, x, y) has a *straddle point* if there exists $a \le t \le b$ such that

$$\|x - u_0\| = \epsilon(x - u_0)(t), \quad \|y - u_0\| = -\epsilon(y - u_0)(t),$$
$$\|(x - u_0, y - u_0)\| = \frac{\epsilon}{|(1, 1)|'}(x - y)(t), \quad \text{where } \epsilon = \pm 1.$$

This definition generalizes those given in [5] and [8] for the special cases in which $E \times E$ is endowed with the maximum and the sum norm respectively.

Note that if (u_0, x, y) has a straddle point, then u_0 is a best simultaneous approximation to x, y from L.

For $x \in E$, we set $T[x] = \{t \in [a,b] : \|x\| = |x(t)|\}$. One says that (u,x,y) *partially alternate* n times on $[a,b]$ if there are $n+1$ points $a \leq t_1 < \cdots < t_{n+1} \leq b$ such that $t_i \in T[z_i - u]$, and $\operatorname{sgn}(z_i(t_i) - u(t_i)) = \epsilon^{i+1} \operatorname{sgn}(z_{i+1}(t_{i+1}) - u(t_{i+1}))$, where $\epsilon = \pm 1$, $z_i, z_{i+1} = x$ or y, $i = 1, \ldots, n$, and sgn denotes the sign function [2].

The next theorem gives a necessary and sufficient condition for ordered best simultaneous approximations when the criterium of approximating comes defined by a M-norm with the commutative property, i.e. such that $|(\|v\|, \|w\|)| = |(\|w\|, \|v\|)|$, for every $v, w \in E$. We will also suppose that $|(1,0)| = 1$.

In particular,when the norm in $\mathbb{R}^2$ is the maximum norm, is obtained the alternation theorem of Rémes–Golomb [10, 7]. On the other hand, when the sum norm is considered, this characterization result has been obtained in [8, Theorem 3.1].

THEOREM 2. (Chebyshev alternation theorem) *Let $E \times E$ be endowed with a commutative* M-*norm; L be an n-dimensional Haar subspace with $1 \in L$; x, y distinct points of E such that $x \leq y$ and $u_0 \in L$. Then u_0 is a best simultaneous approximation to x, y from L if and only if at least one of the following three conditions holds.*

(i) *(u_0, x, y) has a straddle point.*

(ii) *There exists $n+1$ points $a \leq t_1 < \cdots < t_{n+1} \leq b$ such that*

$$\|x - u_0\| = u_0(t_i) - x(t_i), \text{ for each even } i \in \{1, \ldots, n\},$$
$$\|y - u_0\| = y(t_j) - u_0(t_j), \text{ for each odd } j \in \{1, \ldots, n\},$$
$$\|(x - u_0, y - u_0)\| = \frac{1}{2}|(1,1)|(\|x - u_0\| + \|y - u_0\|).$$

(iii) *Condition* (ii) *holds for each odd i and each even j.*

Remark. (a) If the unit sphere of the norm $|\cdot|$ has not segments with slope ± 1, the condition (ii) in the previous theorem becomes

(ii′) There exists $n+1$ points $a \leq t_1 < \cdots < t_{n+1} \leq b$ such that

$$\|x - u_0\| = \|y - u_0\| = u_0(t_i) - x(t_i) = y(t_j) - u_0(t_j),$$

for each even i and for each odd j in $\{1, \ldots, n\}$.

Note that, in this case, $P_{d(L\times L)}(x,y)$ coincides with the set of best simultaneous approximations to x,y from L, when $E\times E$ is endowed with the maximum norm. Then the algorithm in [1] for constrution of elements of the relative Chebyshev center may be used.

(b) When the unit sphere of the norm $|\cdot|$ has some segment with slope ± 1, a necessary condition in order that $(u_0,u_0)\in P_{d(L\times L)}(x,y)$ is

$$\|x-u_0\|+\|y-u_0\|=\frac{1}{2}|(1,1)|\inf_{u\in L}\|(x-u,y-u)\|.$$

So a slight modification of the previous algorithm enables us to find best simultaneous approximations.

References

[1] AMIR, D., ZIEGLER, Z., Construction of elements of relative Chebyshev center, in "Approx Theory and Appl.", Proc. Workshop Technion, Haifa 1980, Academic Press, 1 – 11.

[2] BACOPOULOS, A., On approximation by sums of Chebyshev norms, *Num. Math.,* **16** (1970), 243 – 247.

[3] BAUER, F.L., STOER, J., WITZGALL, C., Absolute and monotonic norms, *Num. Math.,* **3** (1961), 257 – 264.

[4] DEUTSCH, F., MASERICK, P.H., Applications of the Hahn-Banach theorem in approximation theory, *Siam Review,* **9** (3) (1967), 516 – 530.

[5] DUNHAM, C.B., Simultaneous Chebyshev approximation of functions on an interval, *Proc. Amer. Math. Soc.,* **18** (1967), 472 – 477.

[6] GARKAVI, A.L., Duality theorems for approximation by elements of convex sets, *Uspehi Mat. Nauk,* **16** (1961), 141 – 145.

[7] GOLOMB, M., On the uniformly best approximation of functions given by incomplete data, M.R.C. Technical Summary Report 121, Univ. Wisconsin, Madison, 1959.

[8] LING, W.L., MCLAUGHLIN, H.W., SMITH, M.L., Approximation of radom functions, *J. Approx. Theory,* **21** (1977), 10 – 22.

[9] PHELPS, R.R., Uniqueness of Hahn–Banach extensions and unique best approximation, *Trans. Amer. Math. Soc.,* **95** (1960), 238 – 255

[10] RÉMES, E., Sur la détermination de polynomes d'approximation de degré donné, *Comm. Soc. Math. Kharkof,* Ser. **4** (10) (1934), 41 – 63.

[11] SINGER, I., "Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces", Springer–Verlag, New York, 1970.