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Directional Uniform Rotundity in Spaces of Essentially Bounded Functions†

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In this paper, we prove a formula for the directional modulus of rotundity of $L_\infty(X)$, where X is a normed space. As a consequence, we obtain a complete description of the uniform rotundity directions of such a space, and generalize to the vector case the corresponding scalar results of R.R. Phelps [1] and V.I. Zizler [3].

Let X be a normed space. As usual B_X and S_X denote respectively, the unit closed ball and the unit sphere.

The space X is said to be *uniformly rotund in the direction* $z \in X$ (in short, UR$\rightarrow z$) if the directional modulus of rotundity

$$(1) \qquad \delta_X(\rightarrow z, \epsilon) = \inf\left\{1 - \left\|x + \frac{\lambda}{2} z\right\| \ : \ x, x + \lambda z \in B_X, \ \|\lambda z\| \geq \epsilon\right\}$$

is strictly positive for every $0 < \epsilon \leq 2$.

Let (T, Σ, μ) be a positive measure space and X a normed space. The function $x\colon T \rightarrow X$ is said to be *simple* if there exist $T_1, \ldots, T_n \in \Sigma$, and $x_1, \ldots, x_n \in X$ such that $x = \sum_{i=1}^n x_i \chi_{T_i}$, where χ_{T_i} is the characteristic function of T_i. The function $x\colon T \rightarrow X$ is defined as *measurable* if, for every finite measurable set F, there exists a sequence of simple functions $\{s_n\}_{n \in \mathbb{N}}$ such that $x\chi_F = \lim_{n\rightarrow\infty} s_n$ almost everywhere [2].

We use $L_\infty(X)$ to denote the space of equivalence classes of measurable functions $x\colon T \rightarrow X$ such that $t \in T \rightarrow \|x(t)\|_X$ is essentially bounded. It is a linear space normed by $\|x\| = \operatorname{ess\,sup}_{t\in T}\{\|x(t)\|_X\}$, where ess sup denotes the essential supremum of the function x.

The main result is a formula for the directional rotundity modulus of $L_\infty(X)$.

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THEOREM 1. *Let $z \in S_{L_\infty(X)}$. Then*

$$\delta_{L_\infty(X)}(\to z, \epsilon) = \operatorname*{ess\,inf}_{t\in T}\{\delta_X(\to z(t), \epsilon\|z(t)\|_X)\}, \quad 0 \le \epsilon < 2, \tag{2}$$

where ess inf *denotes the essential infimum.*

COROLLARY 2. *The rotundity modulus of $L_\infty := L_\infty(\mathbb{R})$ in the direction $\zeta \in S_{L_\infty}$ is*

$$\delta_{L_\infty}(\to \zeta, \epsilon) = \frac{\epsilon}{2} \operatorname{ess\,inf}\{|\zeta|\}, \quad 0 \le \epsilon < 2. \tag{3}$$

Next we provide a complete description of the uniform rotundity directions in the space $L_\infty(X)$.

THEOREM 3. *Let X be a normed space.*

(i) *The space $L_\infty(X)$ is* UR $\to z$, $z \in S_{L_\infty(X)}$, *if and only if*

$$\operatorname*{ess\,inf}_{t\in T}\{\delta_X(\to z(t), \epsilon\|z(t)\|_X)\} > 0 \quad \text{for } 0 < \epsilon \le 2.$$

(ii) *Let X be a* UR *normed space, then $L_\infty(X)$ is* UR $\to z$ *if and only if*

$$\operatorname*{ess\,inf}_{t\in T}\{\|z(t)\|_X\} > 0.$$

When (T, Σ, μ) is a discrete measure space, one has $L_\infty(X) = \ell_\infty(X)$ and ess inf $=$ inf. Then, Theorems 1, 3 and Corollary 2 hold for $\ell_\infty(X)$. Moreover, formula (2) also holds at $\epsilon = 2$. The same results can be obtained for $\ell_\infty(X_i)$ ($\{X_i\}_{i\in I}$ is a family of normed spaces) i.e., the space of functions $x\colon I \to \bigcup_{i\in I} X_i$, such that $x_i \in X_i$, for each $i \in I$, and $(\|x_i\|_i) \in \ell_\infty$, which is a linear space endowed with the norm $\|x\| = \sup_{i\in I} \|x_i\|_i$.

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