EXTRACTA MATHEMATICAE Vol. **10**, Núm. 2, 183 – 188 (1995)

On the Poles of the Local Resolvent

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AMS *Subject Class.* (1991): 47A11

Received November 6, 1995

1. Introduction

Let X be a complex Banach space and let $T \in L(X)$ be a continuous linear operator on X. If T has the Single Valued Extension Property (see the definition in Section 2), then for every $x \in X$ there exists a unique maximal analytic X-valued function $\widehat{x}_T$, the *local resolvent function of* T *at* x, defined on an open set $\rho(x,T)$, which satisfies $(\lambda - T)\widehat{x}_T(\lambda) = x$, everywhere. The set $\sigma(x,T) := \mathbb{C} \setminus \rho(x,T)$ is a compact subset of the spectrum of T called the *local spectrum of* T *at* x.

The problem that we are concerned with in this paper is the characterization of the poles of the local resolvent $\widehat{x}_T$. It is well-known [6], [9] that a complex number α is a pole of the resolvent $(\lambda - T)^{-1}$ of T if and only if $X = N(\alpha - T)^k \oplus R(\alpha - T)^k$ for some positive integer k, where $N(\alpha - T)^k$ and $R(\alpha - T)^k$ are the kernel and the range of $(\alpha - T)^k$, respectively. In this case we have $N(\alpha - T)^{k+l} = N(\alpha - T)^k$ and $R(\alpha - T)^{k+l} = R(\alpha - T)^k$, for every l; i.e., $\alpha - T$ is a *finite-chain operator.*

We show that α is a pole of $\widehat{x}_T$ if and only if there exists a unique decomposition $x = y + z$, with $y \neq 0$, $y \in N(\alpha - T)^k$ and $\widehat{z}_T$ analytic at α. We observe that $\widehat{z}_T$ analytic at α implies that $z \in R(\alpha - T)^k$ for every k, but the existence of the decomposition $x = y + z$, with $y \neq 0$ and $y \in N(\alpha - T)^k$ and $z \in R(\alpha - T)^k$ for some k is not sufficient to imply that $\widehat{x}_T$ has a pole at α, as it is shown by Example 1. As an application, we describe a class of operators, that includes the totally paranormal operators [7], for which the poles of the local resolvent function are always of order one. Finally we introduce the *locally finite-chain operators at* $x \in X$ as those operators T such that $\sigma(T^{n-1}x, T) \neq \sigma(T^n x, T)$ for some positive integer n or $0 \notin \sigma(x,T)$, and show that T is finite-chain if and only if is locally finite-chain at every $x \in X$.

2. Preliminaries

Given an operator $T \in L(X)$, a complex number λ belongs to the *resolvent set* $\rho(T)$ of T if there exists $(\lambda - T)^{-1} =: R(\lambda, T) \in L(X)$. We denote by $\sigma(T) := \mathbb{C} \setminus \rho(T)$ the *spectrum* of T. The *resolvent map* $R(., T) : \rho(T) \to L(X)$ is analytic.

Moreover, given $x \in X$, we say that a complex number λ belongs to the *local resolvent set* of T at x, denoted $\rho(x, T)$, if there exists an analytic function $w : U \longrightarrow X$, defined on a neigbourhood U of λ, which satisfies

$$(\mu - T)w(\mu) = x, \tag{1}$$

for every $\mu \in U$. The *local spectrum set* of T at x is $\sigma(x, T) := \mathbb{C} \setminus \rho(x, T)$. Since w is not necessarily unique, a complementary property is needed to prevent ambiguity.

An operator $T \in L(X)$ satisfies the *Single Valued Extension Property* (hereafter referred to as SVEP) if for every analytic function $h : U \longrightarrow X$ defined on an open U, the condition $(\lambda - T)h(\lambda) \equiv 0$ implies $h \equiv 0$.

If T satisfies the SVEP, then for every $x \in X$ there exists a unique analytic function $\widehat{x}_T$ defined on $\rho(x, T)$ satisfying (1), which is called the *local resolvent function of T at x*.

If $T \in L(X)$ satisfies the SVEP, then it has two basic properties:

$$\sigma(x, T) = \varnothing \Longleftrightarrow x = 0 \tag{2}$$

$$\sigma(x + y, T) \subset \sigma(x, T) \cup \sigma(y, T). \tag{3}$$

See [3], [4] and [5] for further details.

M. Radjabalipour [8, Theorem 2.3] shows the following theorem about the Local Riesz Decomposition.

Theorem 1. *Suppose $T \in L(X)$ has the SVEP and let F_1 and F_2 be two disjoint compact sets. If $x \in X$ satisfies $\sigma(x, T) \subset F_1 \cup F_2$, then there exists a unique decomposition $x = x_1 + x_2$ so that $\sigma(x_i, T) \subset F_i$ $(i = 1, 2)$.*

The following corollary is immediate considering the above theorem and the property (3).

COROLLARY 1. *Assume that $T \in L(X)$ has the SVEP and let $x \in X$. If $\sigma(x,T) = \sigma_1 \cup \sigma_2$, where σ_1 and σ_2 are disjoint closed sets, then there exists a unique decomposition $x = x_1 + x_2$, where $\sigma_j = \sigma(x_j, T)$, ($j = 1, 2$).*

3. POLES OF THE LOCAL RESOLVENT

For $T \in L(X)$, the poles of the resolvent operator $R(\lambda, T)$ have been characterized as follows [9, Theorem V.10.1]:

$$\alpha \text{ is a pole of } R(\lambda,T) \iff \text{there exists a positive integer } k \text{ so that}$$
$$X = N(\alpha - T)^k \oplus R(\alpha - T)^k.$$

We will try to obtain a similar result for the local resolvent function $\widehat{x}_T$.

Taking into account [4, Proposition 1.5] and [2, Remark 1.5] we have

$$\sigma((\alpha - T)x, T) \subseteq \sigma(x,T) \subseteq \sigma((\alpha - T)x, T) \cup \{\alpha\}$$

and from this fact we can easily derive the following chain of inclusions for the local spectra

$$\sigma(x,T) \supseteq \sigma((\alpha - T)x, T) \supseteq \cdots \supseteq \sigma((\alpha - T)^n x, T) \supseteq \ldots. \tag{4}$$

where α is the only point which may make these subsets different. Hence there is at most an inclusion in (4) which is not an equality.

We recall a characterization of the poles of the local resolvent function obtained in [1, Theorem 1].

PROPOSITION 1. *Assume that $T \in L(X)$ has the SVEP and let $x \in X$. Then $\sigma((\alpha - T)^n x, T) \neq \sigma(x,T)$ if and only if α is a pole of $\widehat{x}_T$ of order $\leq n$.*

From this result we obtain the following consequences which we will need later.

COROLLARY 2. *Suppose $T \in L(X)$ has the SVEP and let $x \in X$. Then the following assertions hold:*

(i) *If $\widehat{x}_T$ has a pole of order n in α, then $\alpha \in \sigma_p(T)$, the point spectrum of T.*

(ii) *The local resolvent function $\widehat{x}_T$ has a pole of order n at α if and only if $\alpha \in \sigma((\alpha - T)^{n-1}x, T) \setminus \sigma((\alpha - T)^n x, T)$.*

(iii) *If $\lambda \in \rho(x,T)$ and $y = \widehat{x}_T(\lambda)$, then $\widehat{x}_T$ has a pole of order n at α if and only if so has $\widehat{y}_T$.*

Using Corollary 1 and 2 we obtain our first characterization of the poles of the local resolvent.

THEOREM 2. *Assume that $T \in L(X)$ has the SVEP and let $x \in X$. Then the following assertions are equivalent:*

(i) *α is a pole of $\widehat{x}_T$ of order n.*

(ii) *There exists a unique decomposition $x = y + z$ such that $y \in N(\alpha - T)^n \setminus N(\alpha - T)^{n-1}$, and $\sigma(z, T) = \sigma(x, T) \setminus \{\alpha\}$.*

COROLLARY 3. *Suppose $T \in L(X)$ has the SVEP and satisfies $X = R(T^n) \oplus N(T^n)$. Given $x \in X$ we have that $0 \in \rho(x, T)$ if and only if $x \in R(T^n)$.*

We observe that the condition $\alpha \notin \sigma(z, T)$ in the previous theorem implies $z \in R(\alpha - T)^k$ for every k. However, $x = y + z$, where $y \in N(\alpha - T)^k \setminus N(\alpha - T)^{k-1}$ and $z \in R(\alpha - T)^k$, does not imply that α is a pole of $\widehat{x}_T$ of order k, as shown in the following example.

EXAMPLE 1. Let T be the operator on the Hilbert space $\ell_2(\mathbb{N})$ defined by $T(e_1) = 0$ and $T(e_{n+1}) = e_{n+2}/(n+1)$. It is clear that $\sigma(T) = \{0\}$, hence T satisfies the SVEP. Taking $x := e_1 + e_3 \neq 0$, we have $\sigma(T^n x, T) = \{0\}$ because $T^n x \neq 0$ for every n. Hence 0 is an essential singularity of $\widehat{x}_T$ by Corollary 2. However x has a unique descomposition $x = y + z = e_1 + e_3$, where $y \in N(T)$ and $z \in R(T)$.

Notice that Theorem 2 implies that, if α is a pole of $\widehat{x}_T$ of order n for some $x \in X$, then the operator $\alpha - T$ is not a finite chain operator of order $n - 1$.

Some interesting applications of Theorem 2 are obtained by considering special classes of operators.

In [3], Dollinger and Oberai proved that if $x \in N(\alpha - T)$, then $\sigma(x, T) = \{\alpha\}$. The converse implication is false in general; for example, if T is an injective quasinilpotent operator.

COROLLARY 4. *Suppose $T \in L(X)$ has the SVEP and satisfies $N(\alpha - T) = \{x \in X : \sigma(x, T) \subseteq \{\alpha\}\}$. If $x \in X \setminus \{0\}$ and α is a pole of $\widehat{x}_T$ of order n, then $n = 1$.*

Recall that an operator T is said to be *totally paranormal* if $\|(\lambda - T)x\|^2 \leq \|(\lambda - T)^2 x\| \|x\|$ for all $x \in X$ and for every $\lambda \in \mathbb{C}$.

Totally paranormal operators satisfy the hypothesis of Corollary 4 [7, Corollary 4.8]. Hence the poles of their local resolvent function are of order ≤ 1. In particular this is true for hyponormal and normal operators on a Hilbert space.

Recall that M is a *μ-space of T* if M is an invariant subspace of T and we have $\sigma(x,T) = \sigma(x,T|M)$ for every $x \in M$.

PROPOSITION 2. *Assume $T \in L(X)$ has the SVEP and there exists a positive integer k such that $X = N(\alpha - T)^k \oplus M$, where M is a μ-space of T. If $x \in X$ and α is a pole of $\hat{x}_T$ of order n, then $n \leq k$.*

Now we give another characterization of the poles of the local resolvent function, similar to the following well-known characterization of the points of the local resolvent given in [5, Theorem 2.2].

Let $T \in L(X)$, $x \in X \setminus \{0\}$ and $\alpha \in \mathbb{C}$. Then $\alpha \in \rho(x,T)$ if and only if there exists a number $R > 0$ and a sequence $(x_k)_{k=0}^{\infty} \subset X$ so that

(i) $(\alpha - T)x_0 = x$;

(ii) $(\alpha - T)x_k = x_{k-1}$; for $k \geq 1$,

(iii) $\|x_k\| \leq R^k$, for $k \geq 0$.

THEOREM 3. *Assume $T \in L(X)$ has the SVEP and let $x \in X$. Then $\alpha \in \mathbb{C}$ is a pole of $\widehat{x}_T$ of order $n > 0$ if and only if there exists a number $R > 0$ and a sequence $(x_k)_{k=-n}^{\infty} \subset X$ so that*

(i) $(\alpha - T)x_0 = x_{-1} - x$.

(ii) $(\alpha - T)x_k = x_{k-1}$, *for* $k > -n$ *and* $k \neq 0$; $(\alpha - T)x_{-n} = 0$ *and* $x_{-n} \neq 0$.

(iii) $\|x_k\| \leq R^k$, *for* $k \geq 0$.

Note that this result is also true when T does not satisfy the SVEP.

4. Locally Finite-Chain Operators

We consider the following integer indexes associated to $T \in L(X)$: *the ascent* of T, denoted by $p(T)$, is the smallest non-negative integer p such that $N(T^p) = N(T^{p+1})$, or $p(T) = \infty$ if no such integer exists; and *the descent* of T, denoted by $q(T)$, is the smallest non-negative integer q such that $R(T^q) = R(T^{q+1})$, or $q(T) = \infty$ if no such integer exists. Recall that $T \in L(X)$ is called a *finite-chain* operator if $p(T) < \infty$ and $q(T) < \infty$, hence $p(T) = q(T)$ (see [6]).

In [9, Theorems V.10.1 & V.10.2] it is proved that α is a pole of the resolvent operator if and only if $\alpha - T$ is a finite-chain operator and $\alpha \in \sigma_p(T)$, the point spectrum of T. Note that Corollary 2 is similar to this result, using the following definition.

DEFINITION 1. Let $T \in L(X)$ and $x \in X$. We say that T is *locally finite-chain at* x if there exists a positive integer n such that $\sigma(T^{n-1}x, T) \neq \sigma(T^n x, T)$, or $0 \notin \sigma(x, T)$.

Observe that T is locally finite-chain at x if and only if there exists a nonnegative integer so that $0 \notin \sigma(T^n x, T)$.

In Theorem 4, we show the relation between finite-chain operators and locally finite-chain operators.

THEOREM 4. *Assume $T \neq 0$ has the SVEP. Then T is a finite-chain operator if and only if T is a locally finite-chain operator at x, for every $x \in X$.*

COROLLARY 5. *Suppose $T \in L(X)$ has the SVEP and α is a pole of $\widehat{x}_T$ for every $x \in X$. Then the order of the pole is always less or equal than a fixed integer n.*

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