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A Note on the Dunford-Pettis Property for Quotients of $C(K)$ Spaces, K Dispersed

Ricardo García

Dpto. de Matemáticas, Univ. de Extremadura, 06071-Badajoz, Spain

(Presented by Jesús F.M. Castillo)

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This note is about the Dunford-Pettis property in quotients of $C(K)$ spaces; here, K is a compact dispersed space. A Banach space X is said to have the Dunford-Pettis property (DPP in short) if weakly compact operators defined on X are completely continuous; equivalently: given weakly null sequences (x_n) and (x_n^*) in X and X^*, respectively, $\lim\langle x_n^*, x_n\rangle = 0$. We say that a compact Hausdorff space K is dispersed or scattered if it does not contain any perfect set. Classical examples are ordinal compacts and the Alexandroff compactification of a discrete set.

1. Quotients of $C(K)$ Spaces

Since all $C(K)$ spaces have DPP, the same occurs for K dispersed. However, not all quotients of a $C(K)$ spaces have DPP. In fact, when X is a Banach space not containing c_0 then every continuous map $T\colon C(K) \to X$ is weakly compact. This means that:

Lemma. *A necessary condition for a quotient of a $C(K)$ space to have DPP is to contain c_0.*

When coming to dispersed spaces, more can be said. Recall from [10] that:

Proposition. *A necessary and sufficient condition for a compact Hausdorff space K to be dispersed is that $C(K)$ is c_0-saturated. Equivalenty, not containing ℓ_1.*

With this, we arrive to the main result of this part:

THEOREM 1. *Let K be a compact Haussdorff space. K is dispersed if and only if every quotient of $C(K)$ has DPP.*

Proof. Let E be a subspace of a $C(K)$ space with K dispersed. Since $C(K)$ does not contain ℓ_1, then E does not contain ℓ_1 and Lohman's lifting applies to the short exact sequence $0 \to E \to C(K) \to C(K)/E \to 0$ to lift weakly convergent sequences in $C(K)/E$, to weakly convergent sequences in $C(K)$. The DPP of $C(K)$ is enough to finish.

Conversely, if every quotient of $C(K)$ has DPP then $C(K)$ cannot contain ℓ_1 (otherwise it would have ℓ_2 as quotient). This is enough to yield that K is dispersed. ∎

The preceding argument shows that if $F \subset \{0,1\}^{\mathbb{N}}$ denotes a compact space formed by finite subsets of $\mathbb{N}$ and S_F is the Schreier-like space in the sense of [4] (defined as the closure of the finite sequences with respect to the norm

$$\|x\|_F = \sup_{A\in F} \sum_{j\in A} |x_j|).$$

Then $C(F)/S_F$ always has the DPP independently of whether S_F has it or not. Examples of spaces in this class are Schreier space, which has not DPP ([2]); Schachermayer's space, which has DPP ([3]); and many others (see [4]).

An even stronger property is the hereditary DPP ($DPPh$, in short): every subspace has DPP. Typical examples of spaces with this property are c_0 and ℓ_1. One has:

LEMMA. ([7]) *$C(K)$ has $DPPh$ if and only if K is dispersed and $K^{(\omega)} = \emptyset$ (where $K^{(\omega)} = \cap_{n\in\mathbb{N}} K^{(n)}$; $K^{(0)} = K$ and $K^{(n)}$ is the set of all accumulation points of $K^{(n-1)}$ for $n \in \mathbb{N}$).*

One moreover has :

THEOREM 2. *Every quotient of $C(K)$ has $DPPh$ if and only if $C(K)$ has $DPPh$.*

Proof. Given Z a subspace of $C(K)/E$ there is M a subspace of $C(K)$ containing E such that $M/E = Z$. Then M is c_0-saturated and the schema of proof of Theorem 1 applies.

Remark. It is an open question to know when every quotient of a $C(K)$ space, K dispersed, is c_0-saturated. Recall that c_0-saturated is not enough to guarantee $DPPh$ (examples: $C(\alpha)$, α compact ordinal greater than ω^ω; Schreier space, etc ...).

2. Dunford-Pettis Property and Duality

It is an open problem to know which conditions for a Banach space X ensures that if X has DPP then X^* also has DPP. Two results are available: if X^* has DPP then X has DPP (obvious); and if X does not contain a copy of ℓ_1 then X has DPP implies that X^* is Schur (via Rosenthal's lemma).

If the club of spaces such that every subspace has DPP has few members, the club of spaces such that all their duals have DPP is still less crowded. Since L_∞- and L_1 -spaces have DPP, and the dual of an L_∞ (resp. L_1)-space is an L_1 (resp. L_∞)-space, those are the first spaces such that every dual has DPP. A second group of spaces such that all duals have DPP was provided by Bourgain ([1]); $C(K, L_1)$, $L_1(\mu, C(K))$ and further iterations. From this, and the fact that L_∞-spaces (resp. L_1-spaces) have bidual complemented in some $C(K)$ (resp. $L_1(\mu)$-space) it follows (see [6] for details) that if E is an L_∞-space then $L_1(\mu, E)$ and all its duals have DPP.

Here we simply remark that given a $C(K)$ space then either K is dispersed, in which case the quotient $C(K)/E$ has DPP for all subspaces E; or K is not dispersed in which case $C(K)$ contains ℓ_1 and then $C(K)$ has ℓ_2 as quotient. If E is a subspace of $C(K)$ such that $C(K)/E$ is reflexive (equivalently, the dual does not contain ℓ_1 (Kadec-Pelczynski [8])). Then the exact sequence

$$0 \to E \to C(K) \to R \to 0$$

induce the exact sequence

$$0 \leftarrow E^* \leftarrow L_1 \leftarrow R^* \leftarrow 0$$

and since R^* does not contain ℓ_1 and L_1 has DPP then E^* has DPP. The bidual sequence

$$0 \leftarrow E^{***} \leftarrow L_1 \leftarrow R^* \leftarrow 0$$

is again exact, and thus E^{***} has DPP. For the same reason, all duals of E have DPP. The basic idea for this "method" was inspired by Kislyakov ([9]).

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