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Some Properties of DF-dF-Spaces

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The objective of this article is to study some properties of DF-dF-spaces. First of all it is proved that Krein-Smulian Property enters into DF-spaces under suitable situation. It has been established that if a DF-dF-space admits an absolute base, then it has to be a Montel Schwartz Köthe space. Next, we observe that a DF-space $\Lambda(P)$ is dF iff bounded sets of $\Lambda(P)$ are precompact. It is shown that the unit vectors form a Schauder basis for the associated sequence space with respect to the weak Schauder basis of a DF-dF-space. For a DF-dF-Köthe space $\Lambda(P) \subseteq \ell^1$, an infinite dimensional dB-space refuses to admit a fully-$\Lambda(P)$-basis. A brief discussion about diametral dimension in the bornology associated with a DF-dF-space appears at the end of the paper.

For the subject of Frechet spaces and DF-spaces we follow [7] and [1], while for Schwartz and nuclear spaces we turn to [13] and [17]. However, concerning various aspects of dF-spaces we refer to [2] and [3].

1. DF-dF-Spaces

In this Section we prove that the Krein-Smulian property holds in a polar-semi reflexive DF-Schwartz space. Besides, that the gap between a DF-space and a dF-space can be filled, to a certain extent, via Schwartz spaces.

Following [2], we say that a l.c. TVS E is a dF-space if it is polar-reflexive and there exists a countable collection $\mathscr{K}$ of compact subsets of E such that if C is a compact subset of E, then there exists a $K \in \mathscr{K}$ such that $C \subseteq K$.

These dF-spaces display the same properties as Fréchet spaces but unlike Fréchet spaces they are always distinguished. Not only dF-spaces have nicer behaviour than the DF-spaces introduced by Grothendieck, but also they share some of the key properties enjoyed by Fréchet spaces (for instance, Krein-Smulian property and Ptak's property) (cf. [2], [3]).

First of all, let it be known that the concept of dF-space introduced by Brauner and Grothendieck's DF-space are incomparable:

EXAMPLE 1.1. Let $E[\tau]$ be an infinite dimensional Banach space and let $E'[\tau_c]$ be its dual endowed with the topology of precompact convergence. The space $E'[\tau_c]$ is a dF, non quasi-barrelled space and it is not a DF-space; it is, however, a gDF and B-complete Schwartz space. Obviously, E itself is a DF space which is not dF since dF-spaces are semi-Montel in view of [2, Proposition 1.8].

The discussion is initiated by the following result. It plays a crucial role in the development of the 1st part of this article.

LEMMA 1.2. (a) *E is polar semi-reflexive DF and Schwartz $\Leftrightarrow$ E is DF and Montel $\Leftrightarrow$ E is DF semi-Montel.*
(b) *If E is DF semi-Montel then E is separable.*

Proof. [17, Proposition 1.4.10] asserts that DF Schwartz spaces are infrabarrelled. Since polar semi-reflexive Schwartz spaces are semi-Montel by [3, Proposition 1.1], polar semi-reflexive DF-Schwartz spaces are Montel. Conversely, DF-Montel spaces are always Schwartz by [17, Proposition 1.4.11]. Now (b) follows from [12, (6), p. 371]. ∎

We have the following proposition which considered per se, speaks to which extent the gap between the DF-spaces and dF-spaces can be bridged via Schwartz spaces.

PROPOSITION 1.3. *A l.c.TVS E is infrabarrelled dF iff E is a polar semi-reflexive DF-Schwartz space.*

Proof. Immediate consequence of Lemma 1.2 and [2, Proposition 1.8]. ∎

EXAMPLE 1.4. We are au fait that Fréchet spaces have the Krein-Smulian Property (cf. [7], [12]). But incomplete normed spaces (which are bornological DF-spaces) are unlikely to have Krein-Smulian property. However, Proposition 1.3 underscores that polar-semi reflexive DF-Schwartz spaces admit Krein-Smulian property in view of [2, Proposition 1.5]. Further, as DF-co Schwartz spaces are Schwartz spaces by [17, Proposition 1.4.11], Krein-Smulian property holds into polar-semi reflexive DF-co Schwartz spaces.

A closed scrutiny of the proof of Proposition 1.3 and the discussion carried out by Horvath (cf. [7, p. 247]) enable us to assert that infrabarrelled dF-spaces are precisely those DF-Schwartz spaces with Krein-Smulian property. Indeed, we have

THEOREM 1.5. *A locally convex space E is an infrabarrelled dF-space iff E is a DF-Schwartz space with one of the following properties:*

(i) *E is complete*

(ii) *E is hypercomplete*

(iii) *E is a Ptak space*

(iv) *E satisfies Krein-Smulian property.*

EXAMPLE 1.6. At this stage it will be interesting to know that there are complete Schwartz spaces which are also co-Schwartz but fail to be Frechet or DF. For an illustration consider the space $\mathscr{D}$ found in [13] (cf. p. 101, Ex. 6.2.4). In fact, $\mathscr{D}^*_\beta$ is an ultrabornological nuclear space.

2. DF-dF-SPACES WITH BASES

This Section 2 identifies topologically a DF-dF-space having an absolute basis with a Montel Schwartz sequence space.

Infrabarrelled dF-spaces are nothing but Schwartz Köthe sequence spaces in case they admit absolute bases, as established by the following;

PROPOSITION 2.1. *Let E be an infrabarrelled dF-space with an absolute basis $\{x_i, f_i\}$. Then E can be topologically identified with a Montel Schwartz Köthe space.*

This result can be proved in two different ways. Recall that given a l.c.s. E, $\mathscr{D}_E$ denotes the set of all continuous seminorms on E.

Proof. (I) Direct consequence of Proposition 1.3 and [8, Theorem 8, p. 314]. ∎

Proof. (II) Since dF-space are complete, by [13, Theorem 10.1.4], we can identify E topologically with the Köthe space $\Lambda(P)$:

$$P = \{p(x_i) \ : \ p \in D_E\}.$$

Thus, $\Lambda(P)$ is a Montel Schwartz space by Lemma 1.2 and Proposition 1.3. ∎

A DF-space which is also a dF-space is called a DF-dF-space.

Since separable DF-spaces are infrabarrelled (cf. [12, p. 399]), one has

COROLLARY 2.2. *Let E be a DF-dF-space with an absolute basis. Then E can be topologically identified with a Montel Schwartz Köthe space.*

Remark 2.3. Observe that by virtue of Lemma 1.2 and Proposition 1.3 a DF-dF-space is separable. Consequently, DF-dF-spaces are Schwartz spaces.

In the following situation, DF-dF-spaces are obtained from co-Schwartz spaces:

PROPOSITION 2.4. *Suppose E is a polar-semi reflexive DF-co Schwartz space. Then E is a DF-dF-space.*

Note. Notice that since DF-co Schwartz spaces are Schwartz spaces, one can also apply directly Proposition 1.3 to obtain the dF-character of E.

This immediately leads to

COROLLARY 2.5. *Each DF-nuclear space is a DF-dF-space.*

Proof. DF-nuclear space are co-nuclear (cf. [13]) and hence co-Schwartz. So, Proposition 2.4 applies because DF-nuclear spaces are reflexive (hence in particular, polar-semi reflexive). ∎

EXAMPLE 2.6. The strong dual of all the Frechet-nuclear spaces found in [13] are DF-dF-spaces. For instance, $(\mathscr{E}(\Delta))^*_\beta$, $(\mathscr{E}_0(\Delta))^*_\beta$, $\mathscr{E}^*_\beta$ and $\mathscr{S}^*_\beta$ (cf. [13, p. 102]).

Note. A DF-space $\Lambda(P)$ is dF iff bounded sets of $\Lambda(P)$ are precompact.

3. DF-dF-SPACES AND ASSOCIATED SEQUENCE SPACES

This Section says that the unit vectors form a Schauder basis for the associated sequence space with respect to the weak Schauder basis of a DF-dF-space. Further, it adds that the associated sequence space admits generalized (absolute) basis in case the given basis is a generalized (absolute) basis.

For various types of bases discussed in this Section we refer to [5] and [11]. Should the need arise, one may be requested to consult [13].

We turn to the associated sequence space corresponding to a Schauder basis $\{x_i, f_i\}$ in a l.c.TVS E. Suppose $P = \{p(x_i) : p \in \mathscr{D}_E\}$. Then P is a Köthe set. Correspondingly, we have the associated locally convex space $\Lambda^\infty(P)$;

$$\Lambda^\infty(P) = \{y \in \omega \ : \ \mu_p(y) = \sup\{|y_i|p(x_i)\} < \infty, \forall p \in \mathscr{D}_E\}.$$

In general the unit vectors $\{e_i\}$ is not a Schauder basis for $\Lambda^\infty(P)$. But the unit vectors form a equicontinuous basis for the sequence space $\Lambda^\infty(P)$ if E is a Frechet Schwartz space; (consequence of [9, Proposition 6.8]). Naturally, question arises, when $\{e_i\}$ will be a Schauder basis for $\Lambda^\infty(P)$ if E is a DF-space? The answer is provided by the following:

PROPOSITION 3.1. *Let $\{x_i, f_i\}$ be a weak Schauder basis for a DF-dF-space E. Then $\{e_i, e_i\}$ is a Schauder basis for $\Lambda^\infty(P)$.*

Proof. By Remark 2.3, E is a Schwartz space. So by a result of Terzioglu [16] for each $p \in \mathscr{D}_E$ there corresponds a $q \in \mathscr{D}_E$ such that

$$\left\{\frac{p(x_i)}{q(x_i)}\right\} \in c_0.$$

Thus, in view of [9, Theorem 5.2] $\Lambda^\infty(P)$ is a Schwartz space. Hence the required result follows from [9, Proposition 6.7]. ∎

Remarks 3.2. (i) For any DF-dF-Köthe space $\Lambda(P)$, $\{x_i, f_i\}$ is a Schauder basis for $\Lambda^\infty(P)$.

(ii) In the light of [9, Proposition 6.8] $\{e_i, e_i\}$ is a Schauder basis for $\Lambda^\infty(P)$ provided $\{x_i, f_i\}$ is a weak Schauder basis for a DF-co-Schwartz space E.

The above result paves the way for the following.

COROLLARY 3.3. *Suppose E is a DF-dF-space having an equicontinuous fully-$\Lambda(P_0)$–basis $\{x_i, f_i\}$, where P_0 is a nuclear Köthe set. Then $\{e_i, e_i\}$ is a fully-$\Lambda(P_0)$-basis for $\Lambda^\infty(P)$.*

Proof. First of all, appealing to Proposition 3.1 we find that $\{e_i, e_i\}$ is a Schauder basis for $\Lambda^\infty(P)$. Now take any $y \in \Lambda^\infty(P)$, $p \in \mathscr{D}_E$ and $a \in P_0$ arbitrarily. Then by the well known Grothendieck-Pietsch criterion (cf. [10], [13]) we obtain a $b \in P_0$ with $\{a_i/b_i\} \in \ell^1$. Since $\{x_i, f_i\}$ is a fully-$\Lambda(P_0)$-basis there exists a $g \in \mathscr{D}_E$, $g = g(p, b)$ such that $p(x_i)b_i \leq g(x_i)$, for all i. Now, if $L = \sum_i (a_i/b_i) < \infty$, then the required assertion follows from the following inequality;

$$\begin{aligned}\sum | < y, e_i > | \; p(x_i)a_i &\leq \sup\{|y_i|p(x_i)b_i\} \left(\sum \frac{a_i}{b_i}\right)\\ &\leq L \sup\{|y_i|g(x_i)\}\\ &= L\mu_g(y). \quad \blacksquare\end{aligned}$$

Remarks 3.4. (i) Under the above hypothesis, E is not necessarily nuclear; but in the case of a nuclear G_∞-set P_0 what we find is that $\{e_i, e_i\}$ is a fully-$\Lambda(P_0)$-basis for $\Lambda^\infty(P)$ if $\{x_i, f_i\}$ is a fully-$\Lambda(P_0)$-basis for a locally convex space E because in this case E becomes nuclear (cf. [11] and hence $\Lambda^\infty(P) = \Lambda(P)$).

(ii) In view of [5, Proposition 3.2], even if $\{x_i, f_i\}$ is only an equicontinuous weak $\Lambda(P_0)$-basis for a DF-dF-space E, the conclusion of the aforesaid result holds because bounded sets are simple in the nuclear Köthe space $\Lambda(P_0)$.

A variant of the above result is contained in the following:

PROPOSITION 3.5. *Let $\{x_i, f_i\}$ be an equicontinuous basis for a DF-dF-space E such that for each $p \in \mathscr{D}_E$ there corresponds a $g \in \mathscr{D}_E$ with $p(x_i) \leq g^2(x_i)$ for all i. Suppose $\{p(x_i)\} \in \Lambda(P_0)$ for each $p \in \mathscr{D}_E$, where P_0 is a Köthe set. Then $\{e_i, e_i\}$ is a fully-$\Lambda(P_0)$-basis for $\Lambda^\infty(P)$.*

Proof. Proposition 3.1 asserts that $\{e_i, e_i\}$ is a Schauder basis for $\Lambda^\infty(P)$. Take any $y \in \Lambda^\infty(P)$, $p \in \mathscr{D}_E$ and $a \in P_0$. The required assertion follows from the inequality

$$\begin{aligned}\sum_i | < y, e_i > | \; p(x_i)a_i &\leq \sum |y_i| g^2(x_i)a_i\\ &\leq \sup\{|y_i|g(x_i)\} \left(\sum g(x_i)a_i\right)\\ &= C\mu_g(y),\end{aligned}$$

where $\sum g(x_i)a_i = C < \infty$. $\blacksquare$

Similarly, one can obtain

PROPOSITION 3.6. *Let E be a DF-dF-space with a weak Schauder basis $\{x_i, f_i\}$ such that for each pair of semi-norms p and g there exists a semi-norm $r \in \mathscr{D}_E$, with $p(x_i)g(x_i) \leq r(x_i)$, for all i. Suppose $\Lambda(P_0)$ is a Köthe space such* *that for some* $p \in \mathscr{D}_E$, $\{1/p(x_i)\} \in \Lambda(P_0)$. *Then* $\{e_i, e_i\}$ *is a fully-*$\Lambda(P_0)$*-basis for* $\Lambda^\infty(P)$.

Remarks 3.7. If either in Proposition 3.5 or Proposition 3.6 $\Lambda(P_0)$ is replaced by ℓ^1 then $\{e_i, e_i\}$ will be a fully-$\Lambda(P)$-basis for $\Lambda^\infty(P)$.

4. dB-Spaces

This Section tells that, for a Schwartz Köthe space $\Lambda(P)$ with $\Lambda(P) \subseteq \ell^1$ an infinite dimensional dB-space fails to admit a fully-$\Lambda(P)$-basis (of course, similar is the situation in the case of a Banach space (cf. [11]).

Following [2] we say that a locally convex space E is a dB-space provided E is polar-reflexive and E has a compact subset which absorbs all compact subsets of E.

Example 4.1. The properties of dB-spaces have been investigated in [2] (cf. [4]). dB-spaces are dF-spaces but the converse is false; even there are DF-dF-spaces which fail to be dB-spaces. For an illustration, consider the DF-nuclear space $(\varphi, \eta(\varphi, \omega))$ (cf. [10]) which becomes a DF-dF-space by the Corollary 2.5. However, $(\varphi, \eta(\varphi, \omega))$ can never be a dB-space as nuclear dB-spaces are finite dimensional by [2, Corollary 1.12].

Remarks 4.2. An infinite dimensional dB-space E fails to be a DF-space because otherwise by Remarks 2.3 E would be a infrabarrelled dB-space and hence by [4, Corollary (B)] E becomes finite dimensional. Therefore a DF-space $\Lambda(P)$ can not be a dB-space.

We pass on to show that an infinite dimensional dB-space does not admit a fully-$\Lambda(P)$-basis, for a DF-dF-space $\Lambda(P)$ with $\Lambda(P) \subseteq \ell^1$. This makes use of

Lemma 4.3. *Let E be a sequentially complete space with a fully-$\Lambda(P)$-basis $\{x_i, f_i\}$, where $\Lambda(P)$ is a Schwartz space with $\Lambda(P) \subseteq \ell^1$. Then E is a Schwartz space.*

Proof. By [11, Proposition 3.8], E can be topologically identified with the Köthe space $\Lambda(M)$;

$$M = \{p(x_i)a_i \; : \; p \in \mathscr{D}_E, \; a \in P\}.$$

Now, by the well-known Schock-Terzioglu criterion (cf. [10]) $\Lambda(M)$ is a Schwartz space because $\Lambda(P)$ is so. Thus E is a Schwartz space. ∎

This sets the stage for

PROPOSITION 4.4. *Let* $\{x_i, f_i\}$ *be a fully-*$\Lambda(P)$*-basis for a* dB*-space* E *where* $\Lambda(P)$ *is a* DF-dF*-space with* $\Lambda(P) \subseteq \ell^1$. *Then* E *is finite dimensional.*

Proof. As dB-spaces are complete, the aforementioned result together with Remark 2.3 gives us the Schwartz character of E. But then [4, Proposition (A)] provides that dB-Schwartz spaces are finite dimensional. ∎

Remarks 4.5. Since dB-Schwartz spaces are finite dimensional by [4, Proposition (A)], making use of [3, Proposition 2.9] we conclude that dB-co Schwartz spaces are finite dimensional.

Note. Observe the striking similarity between normed and dB-spaces.

EXAMPLE 4.6. In view of Remarks 4.5 one concludes that $\mathscr{D}(\Omega)$, $\mathscr{D}(K)$, $(\mathscr{D}(\Omega))^*_\beta$ and $(\mathscr{D}(K))^*_\beta$ are not dB-spaces. Analogously, [2, Proposition 1.11 and Corollary 1.12] implies that the space $\mathscr{D}$ considered by Pietsch [13, p. 101] fails to be a dB-space. Similar is the case for its strong dual $\mathscr{D}^*_\beta$.

5. DIAMETRAL DIMENSION IN BORNOLOGY

This is the final Section of the current discussions which incorporates a study on the diametral dimension for Schwartz and nuclear bornologies associated with a dF-space. It has been found that the Schwartz and nuclear bornologies associated with dB and normed spaces are more or less similar in nature. This small Section slightly deviates from the main direction of study, namely, properties of DF-dF-spaces.

The concept of diametral dimension appeared for the first time in the works of Bessaga, Pelczynski and Rolewicz [1]. From the bornological point of view we have the notions of Δ-and γ-diametral dimension of a c.b.s. (complete bornological space) (cf. [6]).

DEFINITION 5.1. Suppose E is a c.b.s. We call Δ-diametral dimension (resp. Γ-diametral dimension) of E, denoted by $\Delta_{\mathbb{B}}(E)$ (resp. $\Gamma_{\mathbb{B}}(E)$), the collection of all sequence of non-negative numbers (a_i) wich satisfy the following condition:

For each $A \in \mathbb{B}(E)$, there exists $B \in \mathbb{B}(E)$, $A < B$, such that $a_i\delta_i(A, B) \to 0$ (resp. $a_i\gamma_i(A, B) \to 0$), where δ_i is the Kolmogorov's diameter and γ_i is the i-th section of A with respect to B (cf. [6]).

If E is a quasi-complete locally convex space the collection of all bounded sets of E (in von Neumann sense) forms a bornology on E. Endowed with this bornology, E is a c.b.s. denoted by ${}^bE$.

The Δ-diametral codimension (resp. Γ-diametral codimension) of a l.c.TVS E is the Δ-diametral dimension (resp. Γ-diametral dimension) of ${}^bE$, denoted respectively by $\Delta_{\mathbb{B}}({}^bE)$ and $\Gamma_{\mathbb{B}}({}^bE)$.

DEFINITION 5.2. Let E be a l.c.TVS. We call Δ-diametral dimension (resp. γ-diametral dimension) of E, denoted by $\Delta_{\mathscr{U}}(E)$ (resp. $\Gamma_{\mathscr{U}}(E)$), the collection of all sequences of non-negative numbers (a_i) satisfying:

For each $u \in \mathscr{U}_E$ there exists a $\nu \in \mathscr{U}_E$, $\nu < u$ such that $a_i\delta_i(\nu, u) \to 0$ (resp. $a_i\gamma_i(\nu, u) \to 0$).

A c.b.s. E is said to be Schwartz (resp. nuclear) if each disk $A \in \mathbb{B}(E)$ is absorbed by a disc $B \in \mathbb{B}(E)$ in such a way that the canonical embedding $\Phi_{A,B}$ is compact (resp. nuclear). A quasi-complete l.c.TVS E is Co-nuclear if ${}^bE$ is nuclear and E is Co-Schwartz if ${}^bE$ is Schwartz .

Because of Proposition 1.3. Schwartz spaces bridge the gap between DF and dF spaces. So we confine our attention to the study of Schwartz (nuclear) bornologies associated with dF-spaces.

Remarks 5.3. Since infrabarrelled dF-spaces are Schwartz spaces by Proposition 1.3, it follows from a well-known result (cf. [15]) that

$$(1, 1, \dots) \in \Delta_{\mathscr{U}}(E)$$

for an infrabarrelled dF-space E. However, if it is a bornological dF-space we have;

PROPOSITION 5.4. *Let E be a bornological dF-space. Then* $(1, 1, \dots) \in \Delta_{\mathbb{B}}(E)$.

Proof. E is a c.b.s. as dF-spaces are complete. In addition, E is Schwartz by Proposition 1.3 being infrabarrelled dF. Now [6, Theorem 10(a)] applies and the conclusion follows. ∎

Note. For a DF-dF-space E; $(1, 1, \dots) \in \Delta_{\mathscr{U}}(E)$.

Remarks 5.5. If a dF-space E is co-Schwartz, then making use of [6, Corollary 2(a)] we find that

$$(1, 1, \dots) \in \Gamma_{\mathscr{U}}(E^*_\beta)$$

because E is a complete semi-reflexive space. Consequently, we infer that the following holds in view of [6, Corollary 1(a)];

PROPOSITION 5.6. *Let E be a bornological dF-space. Suppose E is Co-Schwartz, then*

$$(1,1,\dots) \in \Delta_{\mathbb{B}}(E) \cap \Gamma_{\mathscr{U}}(E^*_\beta) \cap \Delta_{\mathbb{B}}({}^bE).$$

For a dB-space we have;

PROPOSITION 5.7. *Let E be an infinite dimensional dB-space. Then*

$$(1,1,\dots) \notin \Delta_{\mathbb{B}}(E) \cup \Gamma_{\mathscr{U}}(E^*_\beta) \cup \Delta_{\mathbb{B}}({}^bE).$$

Proof. If $(1,1,\dots) \in \Delta_{\mathbb{B}}(E)$, then E is Schwartz (cf. [15]), but then dB-Schwartz spaces are finite dimensional by [4, Proposition (A)]. On the other hand, if $(1,1,\dots) \in \Gamma_{\mathscr{U}}(E^*_\beta)$ then [6, Corollary 2(a)] implies that E is co-Schwartz. But, by Remarks 4.5, dB-co Schwartz spaces are finite dimensional. Similar is the case if $(1,1,\dots) \in \Delta_{\mathbb{B}}({}^bE)$; of course here we use [6, Corollary 1(a)]. ∎

Comming to dF-nuclear space, we find that the situation is slightly different, as borne out by the following result.

PROPOSITION 5.8. *Let E be bornological dF-nuclear space. Then, for each positive number k,*

$$(1,2^k,3^k\dots) \in \Delta_{\mathbb{B}}(E) \cap \Gamma_{\mathscr{U}}(E^*_\beta) \cap \Delta_{\mathbb{B}}({}^bE).$$

Proof. The first part namely, $(1,2^k,3^k\dots) \in \Delta_{\mathbb{B}}(E)$ follows from [6, Theorem 10(b)] while the other part, that is, $(1,2^k,3^k\dots) \in \Gamma_{\mathscr{U}}(E^*_\beta)$ is a consequence of [6, Corollary 2(b)] as dF-space are semi-reflexive and a dF-space E is nuclear iff E^p is nuclear, by [2, Proposition 1.11]. Besides, $(1,2^k,3^k\dots) \in \Delta_{\mathbb{B}}({}^bE)$ comes from [6, Corollary 1(b)]. ∎

Note. The above result holds for DF-nuclear spaces.

PROPOSITION 5.9. *Let E be a dB-space. Then*

$$(1,2,3,\dots) \notin \Gamma_{\mathscr{U}}(E^*_\beta) \cup \Delta_{\mathbb{B}}({}^bE) \cup \Delta_{\mathscr{U}}(E).$$

Proof. This follows directly from [6, Corollary 1 (b) and Corollary 2(a)], as dB-co nuclear spaces are finite dimensional because dB-co Schwartz spaces are finite dimensional, while dB-nuclear spaces are finite dimensional by [2, Corollary 1.12]. ∎

Note. Observe that Proposition 5.7 and Proposition 5.9 are akin to that of normed spaces.

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