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The Hermitian Intersection Form for a Family of Elliptic Curves

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1. THE MAIN RESULT

Let $\tilde{X}$ be an open polydisc with center 0 in $\mathbf{C}^{n+1}$ and let $\tilde{f}\colon \tilde{X} \longrightarrow \mathbf{C}$ be a holomorphic function such that $\tilde{f}(0) = 0$ with an isolated singularity at 0. Let D denote the unit-disc in $\mathbf{C}$ and $f\colon X \longrightarrow D$ a Milnor representative of $\tilde{f}$; it means that X is a Stein and contractible open subset of $\tilde{X}$ and the restriction of f to $X - f^{-1}(0)$ induces a C^∞ locally trivial fibration of $X - f^{-1}(0)$ ([4]). Let $s_0 \in D - \{0\}$ be a base-point and $f^{-1}(s_o)$ the Milnor fibre.

Set $E = H^n(f^{-1}(s_o), \mathbf{C})$ and let

$$j_o : H^n_c(f^{-1}(s_o), \mathbf{C}) \longrightarrow H^n(f^{-1}(s_o), \mathbf{C})$$

denote the natural map (H^n_c denotes the cohomology with compact support) and E^∞ denote the image of E under j_o.

In [1] D. Barlet shows that the intersection form on $H^n_c(f^{-1}(s_o), \mathbf{C})$ induces a bilinear form k on E^∞, called the hermitian intersection form, defined by:

$$\forall a, b \in E^\infty : \quad k(a, b) = (2\pi i)^{-n} \text{ trace } (a \cup \bar{b}),$$

where $\cup$ denotes the cup-product of cohomology classes, and the "trace" denotes the isomorphism:

$$H^{2n}_c(f^{-1}(s_o), \mathbf{C}) \xrightarrow{\sim} \mathbf{C}$$

given by integration on the smooth fibre $f^{-1}(s_o)$. This form k is a convenient tool for the study of singularities of functions obtained by integrating smooth forms with compact support, on the fibers of a given holomorphic function with an isolated critical point. Indeed, k provides many informations on the period application, and is closely related to the hermitian canonical form ([1] and [2]). So, it's a natural task to understand the behaviour of k in relation to a parameter which describes a family of singularities not necessarily with constant Milnor number. Fix $s \in D - \{0\}$ once and for all.

The main result of this work is the following:

THEOREM. *Let (P_λ) be a family of complex polynomials in $\mathbf{C}[x,y]$ of degree 3, such that $P_\lambda(0,0) = 0$ and $(0,0)$ is an isolated singularity of P_λ. Let $s \in D - \{0\}$, and consider the family $(C_{\lambda,s})$ of elliptic curves given by:*

$$P_\lambda(x,y) = s\ .$$

Let μ_λ be the Milnor number of the singularity P_λ, and let $\mathcal{B}_\lambda$ be the multivalued horizontal basis of the Gauss-Manin bundle. Denote by $(a_{i,j}(\lambda))_{1\leq i,j\leq \mu}$ the matrix in $\mathcal{B}_\lambda$ of the hermitian intersection form k. Then, the rank of $\mathcal{B}_\lambda$ is 2, and moreover, each $a_{i,j}(\lambda)$ is either identically zero, or a continuous function such that $a_{i,j}(\lambda) \neq 0$ for $|\lambda|$ small enough.

Proof. For a generic λ (i.e. such that P_λ has an isolated singularity at (0,0)) consider the homogeneous polynomials:

$$h_\lambda(x,y,t) = \tilde{P_\lambda}(x,y,t) - st^3,$$

where $\tilde{P_\lambda}(x,y,t)$ denotes the homogeneous polynomial associated with $P_\lambda(x,y)$. Under a suitable linear change of coordinates in $\mathbf{P^2C}$, and taking into account that P_λ have a critical point at (0,0), we see that the the compactification of the family $(C_{\lambda,s})$ is described by the following equations:

$$h_\lambda(X,Y,T) = Y^2T - 4X^3 - b_\lambda(T,s) = 0, \tag{1}$$

where $b_\lambda(T,s) \in \mathbf{C}[T]$ is homogeneous of degree 3 and satisfy $b_\lambda(1,s) \neq 0$.

The affine curves associated with (1) are given by:

$$y^2 = 4x^3 - b_\lambda(1,s). \tag{2}$$

So, as the modular invariant is 0 for any element of the family, we get that for any $s \in D - \{0\}$ the curve $(C_{\lambda,s})$ is isomorphic to the one whose equation is $y^2 - x^3 = s$. From this, we deduce that the hermitian intersection form for P_λ has the same rank than that of the cusp singularity. But, as the intersection form is nondegenerate (e.g. by Poincaré duality argument), we easily deduce that the rank for the cusp is 2.

Now, to study the coefficients $a_{i,j}(\lambda)$, we first look at each $(C_{\lambda,s})$ as the elliptic curve associated to a lattice $\Lambda_{\lambda,s}$. The equation (2) becomes:

$$y^2 = 4x^3 - g_3(\Lambda_{\lambda,s}),$$

where $g_3(\Lambda_{\lambda,s})$ is the classical second elliptic invariant.

For each point (x, y) of $(C_{\lambda,s})$, there exists one and only one $z \in \mathbf{C}$ (modulo $(\Lambda_{\lambda,s})$) such that $\wp'(z) = y$ and $\wp(z) = x$.

Then, the description of $a_{i,j}(\lambda)$ will result from the study of the following integrals:

$$\int_{\gamma_p} w_q, \quad 1 \leq p, q \leq \mu_\lambda,$$

where w_q are suitable rational d-closed 1-forms (see II.2) below) and γ_p are 1-cycles in the evanescent homological basis of the singularity P_λ ([3]).

Thus, we are led to study the integrals

$$\int_{\varepsilon}^{\omega_\lambda+\varepsilon} F_\lambda(\wp(z), \wp'(z))\, dz, \quad (|\varepsilon| \text{ small enough}),$$

where F_λ are rational functions with respect to $\wp(z)$ and $\wp'(z)$.

Finally, using suitable results on elliptic integrals (e.g. [8]), we get that each $a_{i,j}(\lambda)$ is either zero or is a continuous function outside the discriminant of P_λ. This completes the proof of theorem. ▮

N.B. : In practice, the functions $a_{i,j}(\lambda)$ are always explicitly gotten. The example below gives the detailed steps of the study.

2. Application to the family : $x^3 + y^3 + \lambda x^2 y = s$

This example is a fundamental one because its study requires to make explicit all the ingredients needed for the proof of the general case.

Consider the family of elliptic curves $X_\lambda(s)$ given in $\mathbf{C}^2$ by:

$$x^3 + y^3 + \lambda x^2 y = s,$$

where $s \in \mathbf{C}^*$ and λ is a complex parameter such that $4\lambda^3 + 27 \neq 0$.

1) Uniformization of the curves $X_\lambda(s)$:

For $s \neq 0$ and a given λ, the curve $X_\lambda(s)$ is a non-singular plane cubic with genus 1, so it's an elliptic curve. To get its uniformization we first turn its equation into the Weierstrass canonical form. Following [7, Chap. III] we have to choose a flex point I and a coordinate system in which, on one hand, the tangent line to $X_\lambda(s)$ at I coincides with the tangent line at infinity, and on the other hand, the point I is sent at infinity on the y-axis. The nine flex points of $X_\lambda(s)$ are gotten by intersecting the compact curve $\overline{X_\lambda(s)}$ of $\mathbf{P^2C}$ with its hessian. An easy computation shows that one can take $I = (1 : \alpha : 0)$ where α is the real root of $z^3 + \lambda z + 1$ (for simplicity we assume $\lambda \in \mathbf{R}$).

The desired change of homogeneous coordinates is then given by

$$\begin{cases} t' = (3 + 2\lambda\alpha)\, x + (\lambda + 3\alpha^2)\, y \\ y' = y \\ x' = t \end{cases} .$$

Putting

$$Y = y' + \frac{1}{2}\, \frac{\lambda\sigma - 3\theta}{3\theta^2 - 2\lambda\sigma\theta} \quad \text{and} \quad X = \left(\frac{s\sigma^3}{4(3\theta^2 - 2\lambda\sigma\theta)}\right)^{\frac{1}{3}} x'$$

we easily get for $X_\lambda(s)$ the Weierstrass canonical form

$$Y^2 = 4X^3 + \frac{1}{2\lambda\sigma\theta - 3\theta^2} + \frac{1}{4}\left(\frac{\lambda\sigma - 3\theta}{3\theta^2 - 2\lambda\sigma\theta}\right)^2, \tag{*}$$

where $\sigma = 3 + 2\lambda\alpha$ and $\theta = \lambda + 3\alpha^2$ (for the complex cubic root we choose the principal determination of the logarithm). We know then that there exits one and only one $z \in \mathbf{C}$ (modulo a lattice Λ described below) such that $\wp'(z) = Y$ and $\wp(z) = X$ where $\wp$ denotes the Weierstrass elliptic function.

If we denote by ω_λ and ω'_λ the periods of $X_\lambda(s)$, we have

$$g_2\,(\omega_\lambda, \omega'_\lambda) = 0 \quad (g_2 \text{ denotes the first Weierstrass invariant})$$

and thus

$$\frac{\omega'_\lambda}{\omega_\lambda} = j = \exp\left(\frac{2\pi i}{3}\right) \quad \text{(see e.g. [5, Chap. VII]).}$$

Remark: $4\lambda^3 + 27 \neq 0 \Longleftrightarrow 3\theta^2 - 2\lambda\sigma\theta \neq 0$.

2) The multivalued horizontal basis of the Gauss-Manin bundle:

If $g\colon \mathbf{C}^{n+1} \longrightarrow \mathbf{C}$ is any holomorphic function with an isolated singularity at 0, we call the Gauss-Manin bundle of g, the vector bundle defined on $D - \{O\}$ by

$$\bigcup_{s \in D - \{O\}} H^n(g^{-1}(s), \mathbf{C})$$

and the following projection

$$s \longmapsto H^n(g^{-1}(s), \mathbf{C}),$$

where $g^{-1}(s)$ denotes the Milnor fibre of g (see [3] for more details).

Consider now the family of polynomials $f_\lambda(x,y) = x^3 + y^3 + \lambda x^2 y$. Since f_λ are homogeneous polynomials, we know from [4] that their Milnor fibre is homeomorphic to the corresponding affine fibre. For any fixed λ, the function f_λ has an isolated singularity at 0. Denote $J(f_\lambda)$ the jacobian ideal of f_λ and by $\mathbf{C}\{x,y\}$ the algebra of complex convergent series in x and y. We call Milnor number of the singularity f_λ the complex dimension μ of the artinian local algebra $\mathbf{C}\{x,y\}/J(f_\lambda)$. Clearly, the classes of $1, x, x^2, y$ determine a $\mathbf{C}$-basis for the local algebra, so we get $\mu = 4$. Set $w = xdy - ydx$, an easy computation gives

$$dw = \frac{2}{3}\frac{df_\lambda}{f_\lambda} \wedge w, \quad d(xw) = \frac{df_\lambda}{f_\lambda} \wedge (xw),$$

$$d(yw) = \frac{df_\lambda}{f_\lambda} \wedge (yw), \quad d(x^2 w) = \frac{4}{3}\frac{df_\lambda}{f_\lambda} \wedge (x^2 w).$$

Then, we know from [1] that the set $\mathcal{B}$ of cohomology classes:

$$\mathcal{B} = \left\{ \left[f_\lambda^{-\frac{2}{3}} w\right], [f_\lambda^{-1} xw], [f_\lambda^{-1} yw], \left[f_\lambda^{-\frac{4}{3}} x^2 w\right] \right\}$$

provides the desired multivalued horizontal basis of the Gauss-Manin bundle.

3) Description of the hermitian intersection form for the family $X_\lambda(s)$:

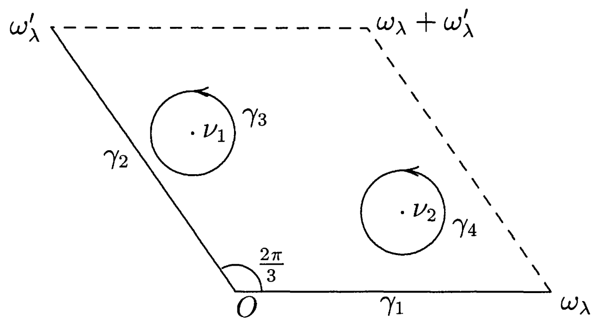
a) Matrix of k in the standard $\mathbf{C}$-basis of homology:

Taking into account the results of 2) we get that, as complex space, $X_\lambda(s)$ is the quotient of $\mathbf{C}$ by the following lattice:

$$\Lambda = Z + jZ, \quad \text{where } j = \exp\left(\frac{2i\pi}{3}\right).$$

Moreover, the homology group $H_1(X_\lambda(s), \mathbf{C})$ has a basis $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ made up of the so-called evanescent cycles ([3]).

In our situation, the cycles γ_1 and γ_2 are respectively identified with $[0, \omega_\lambda]$ and $[0, \omega'_\lambda]$ in $\mathbf{C}/\Lambda$. The cycles γ_3 and γ_4 are represented by the two loops below:



The complex numbers ν_1 and ν_2 are defined modulo Λ by

$$\begin{cases} \wp(\nu_i) &= a_i, \\ \wp'(\nu_i) &= b_i, \end{cases}$$

where a_i and b_i are respectively the image of $(1 : \sigma_1 : 0)$ and $(1 : \sigma_2 : 0)$ under the coordinate transformation used in a) (σ_1 and σ_2 denote the roots of $z^3 + \lambda z + 1$ distinct from α).

The description we have gotten for the evanescent basis gives the following:

LEMMA. *In the homolological basis $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$, the matrix of the intersection form is*

$$A = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

b) Matrix of k in the multivalued horizontal basis:

With the notations of the previous paragraph, we get, modulo d-exact differential forms

$$w = 2xdy, \quad xw = \frac{3}{2}\,x^2dy, \quad x^2w = \frac{4}{3}\,x^3dy \quad \text{and} \quad yw = 3xydy.$$

Set

$$w_1 = w, \quad w_2 = x^2w, \quad w_3 = xw \quad \text{and} \quad w_4 = yw.$$

The matrix of change of basis, from the standard homological basis to the dual one, is

$$g = i(\phi_{p,q})_{1\le p,q\le 4} \text{ where } \phi_{p,q} = \int_{\gamma_p} w_q.$$

In the multivalued horizontal basis $\mathcal{B}$,the matrix K of k is

$$K = g^t.\, A.\, \bar{g},$$

where g^t denotes the transpose matrix and $\bar{g}$ the complex conjugation.

Taking into account the configuration of the matrix A, to describe K we have to compute the period integrals $\phi_{p,q}$ with $1 \le p \le 4$ and $1 \le q \le 2$. We get:

LEMMA. *Let ω_λ and ω'_λ denote the periods of the elliptic curve $X_\lambda(s)$. We have*

$$\phi_{1,1} = F\,\omega_\lambda \text{ and } \phi_{2,1} = j\;\phi_{1,1},$$

$$\phi_{1,2} = \frac{G}{\omega_\lambda} \text{ and } \phi_{2,2} = j^2\phi_{1,2},$$

$$\phi_{3,1} = H\,\omega_\lambda \text{ and } \phi_{3,2} = j\;\phi_{3,1},$$

$$\phi_{4,1} = \phi_{4,2} = 0,$$

where F, G and H are constants depending on λ and whose explicit expressions are given below.

Proof. The method of calculation is almost the same for all the period integrals, so, we restrict ourselves to $\phi_{1,1}$ and $\phi_{2,1}$.

Let X, Y the coordinates into which the equation $x^3 + y^3 + \lambda x^2 y = s$ of $X_\lambda(s)$ is written under the Weierstrass canonical form. With notations of 1), set

$$C = \frac{1}{2}\,\frac{\lambda\sigma - 3\theta}{3\theta^2 - 2\lambda\sigma\theta} \quad\text{and}\quad D = \left(\frac{s\sigma^3}{4(3\theta^2 - 2\lambda\sigma\theta)}\right)^{\frac{1}{3}}.$$

Since $(3\theta^2 - 2\lambda\sigma\theta)\sigma \ne 0$,we have $D \ne 0$, and a direct computation gives

$$dy = \frac{1}{D}\,\frac{dY}{X} - \frac{Y - C}{D}\,\frac{dX}{X^2}$$

and, we get successively

$$\phi_{1,1} = \int_{\gamma_1} w_1 = \int_{\gamma_1} 2x\,dy = \frac{2}{D^2\sigma}\,(I_1 + I_2).$$

But

$$I_1 = (1+C)\int_{\gamma_1} \frac{dY}{X^2} \ - \theta \int_{\gamma_1} \frac{Y dY}{X^2},$$

so, using the fundamental relation $\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3$ and its derivative , we get

$$\int_{\gamma_1} \frac{dY}{X^2} = \int_{\varepsilon}^{\omega_\lambda+\varepsilon} \frac{\wp''(z)}{\wp^2(z)}\, dz \ = \ 6\ \omega_\lambda$$

(ε will denote a complex number with $|\ \varepsilon\ |$ small enough). And

$$\int_{\gamma_1} \frac{Y dY}{X^2} \ = \ \int_{\varepsilon}^{\omega_\lambda+\varepsilon} \frac{\wp'(z)\ \wp''(z)}{\wp^2(z)}\ dz \ = \ \int_{\varepsilon}^{\omega_\lambda+\varepsilon} \wp'(z) dz \ = \ 0.$$

Hence,

$$I_1 = 6\ (1+C)\ \ \omega_\lambda.$$

On the other hand, we have

$$I_2 = (1+C)\ J_1 - (1+C+C\theta)\ J_2 + \theta\ J_3,$$

where

$$J_1 = \int_{\gamma_1} \frac{dX}{X^3} \ = \ \int_{\varepsilon}^{\omega_\lambda+\varepsilon} \frac{\wp'(z)}{\wp^3(z)}\ dz \ = \ 0,$$

$$J_2 = \int_{\gamma_1} \frac{Y}{X^3}\ dX = \int_{\varepsilon}^{\omega_\lambda+\varepsilon} \frac{\wp'^2(z)}{\wp^3(z)}\ dz = \frac{1}{2}\int_{\varepsilon}^{\omega_\lambda+\varepsilon} \frac{\wp''(z)}{\wp^2(z)}\ dz \ = \ 3\ \omega_\lambda$$

and

$$J_3 = \int_{\gamma_1} \frac{Y^2}{X^3} dX \ = \ \int_{\varepsilon}^{\omega_\lambda+\varepsilon} \frac{\wp'^3(z)}{\wp^3(z)}\ dz$$

$$= \ [\ \frac{\wp'^2(z)}{2\wp^2(z)}\ ]_{\omega_\lambda+\varepsilon}^{\varepsilon} + 6\int_{\varepsilon}^{\omega_\lambda+\varepsilon} \wp'(z)\ dz \ = 0.$$

Thus

$$I_2 = -3\ (1+C+C\theta)\ \ \omega_\lambda.$$

Consequently, we get

$$\phi_{1,1} \ = \ F\,\omega_\lambda \ \text{ where } \ F \ = \ \frac{(1+C-C\theta)}{D^2\sigma}\ \ \omega_\lambda.$$

Now, if we substitute ω'_λ for ω_λ in the previous results and if we recall that $\omega'_\lambda = j\ \omega_\lambda$, then we easily deduce the announced result for $\phi_{2,1}$. In the same way, we get

$$G = \frac{24}{D^4\sigma^3\sqrt{3}}\ (1+C-C\theta)\ \ \left(27\theta^2 - \frac{(1+C)^2}{g_3(\omega_\lambda, \omega'_\lambda)}\right).$$

N.B. : From the identity (*) of 1) we directly deduce the value of $g_3(\omega_\lambda, \omega'_\lambda)$. Moreover we have

$$g_3(\omega_\lambda, \omega'_\lambda)) = \omega_\lambda^{-6}\, g_3(1,j) \quad ([5])$$

and

$$g_3(1,j) = \frac{1}{(2\pi)^6}\, \Gamma^{18}(\frac{1}{3}) \quad ([8])$$

so, we obtain the exact value of the periods.

And finally we get

$$H = \frac{9\,(C + C^2)\theta}{D^2\sigma}.$$

Hence, the coefficients F, G, H are totally described, and the proof of the lemma is complete. ▮

We prove now the folowing:

PROPOSITION. *With respect to the multivalued horizontal basis $\mathcal{B}$, the matrix K of the hermitian intersection form k is*

$$K = \sqrt{3}\begin{pmatrix} \mid F\omega_\lambda \mid^2 & 0 & \bar{F}H \mid \omega_\lambda \mid^2 & 0 \\ 0 & -\left|\dfrac{G}{\omega_\lambda}\right|^2 & 0 & 0 \\ F\bar{H}|\omega_\lambda|^2 & 0 & |H\omega_\lambda|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where F, G, H are the constants computed above. Furthermore, these constants depend continuously on the parameter λ, and satisfy $(FGH)(\lambda) \neq 0$ for $\mid \lambda \mid$ small enough.

Proof. a) We easily get K from the relation $K = g^t\, A\, \bar{g}$ and the computation of the integral periods made above. b) The explicit expressions of F, G and H clearly show that these coefficients are continuous functions of λ. c) To prove the last assertion of the theorem, we first show that $(FGH)(0) \neq 0$. For $\lambda = 0$ we get: $\alpha = -1$, $\sigma = \theta = 3$ and $C = -\frac{1}{6}$. But the identity $F(0) = 0$ implies $C = \frac{1}{1-\theta} = \frac{1}{2}$ thus $F(0) \neq 0$. Even so, we get $G(0) \neq 0$ because the identity $G(0) = 0$ would give $C = 0$ or $C = -1$. At last, $H(0) \neq 0$ because the condition $H(0) = 0$ leds to the approximation $\Gamma(\frac{1}{3}) \simeq 1{,}26...$ which is false (we recall that $\Gamma(\frac{1}{3}) \simeq 2,68...$). Hence $(FGH)(0) \neq 0$, and we conclude by continuity. This completes the proof of the Proposition. ∎

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