EXTRACTA MATHEMATICAE Vol. 10, Núm. 2, 115 – 118 (1995)

An Inequality for Norm of Operators†

MARÍA D. ACOSTA

Dpto. de Análisis Matemático, Univ. de Granada, 18071-*Granada, Spain*

(Presented by C. Benítez)

AMS *Subject Class.* (1991): 47A30, 47A05 *Received October* 16, 1995

In this paper we deal with the question wether there is a positive constant K, such that for every X, Y, Z normed spaces, given two elements $T \in L(X,Y)$ (the set of all bounded and linear operators from X to Y) and $S \in L(X,Z)$, the inequality

$$(*) \qquad \|T\|\|S\| \leq K \sup\{\|Tx\|\|Sx\| : x \in X, \|x\| \leq 1\}$$

is satisfied.

In case the answer is affirmative, what is the best choice for the universal constant K?

The first question arised in a recent paper on Jordan-Banach algebras [3], where the authors give an argument to show that there is such a K satisfying $(*)$, by using ultraproducts. Even so, they proved the same result by elementary methods, obtaining not only the existence of the constant, but also an admissible value. In the present note, we will prove that the value $K = 4$ works and this is the best choice for any X, Y, Z and any pair of operators $T \in L(X,Y)$, $S \in L(X,Z)$.

To begin with, we will reduce the question to functionals, instead of operators:

LEMMA. *Let X be a normed space and assume there is $K > 0$ such that for any pair of elements $u^*, v^* \in X^*$ (the space of all bounded and linear functionals on X) it is satisfied*

$$\|u^*\|\|v^*\| \leq K \sup\{|u^*(x)v^*(x)| : x \in X, \|x\| \leq 1\},$$

then K also verifies $(*)$ for any Y, Z normed spaces and any $T \in L(X,Y)$, $S \in L(X,Z)$.

†Research partially supported by D.G.I.Y.T., project no. PB93 – 1142.

Proof. Let us fix normed spaces Y, Z and operators $T \in L(X,Y)$, $S \in L(X,Z)$; now we choose functionals y^* and z^* in the unit balls of Y^* and Z^*, respectively, and apply the hypothesis to the functionals $u^* = T^*(y^*)$ and $v^* = S^*(z^*)$ to get

$$\begin{aligned}\|T^*y^*\|\|S^*z^*\| &\leq K \sup\{|(T^*y^*)(x)||(S^*z^*)(x)| : x \in X, \|x\| \leq 1\}\\ &\leq K \sup\{\|Tx\|\|Sx\| : x \in X, \|x\| \leq 1\}.\end{aligned}$$

Now, since we can argue as above for any functionals y^*, z^* in the unit ball of Y^*, Z^*, respectively, the Hahn-Banach Theorem gives us the desired formula. ∎

Now we are in conditions to state the announced result:

THEOREM. *Let X, Y and Z be normed spaces, $T \in L(X,Y)$ and $S \in L(X,Z)$. Then*

$$\|T\|\|S\| \leq 4 \sup\{\|Tx\|\|Sx\| : x \in X, \|x\| \leq 1\}$$

and 4 is the best possible constant in the above inequality.

Proof. First note that we can clearly assume the normed spaces to be real. In view of the Lemma, it is enough to deal with the case $Y = Z = \mathbb{R}$. So, let us choose two elements $u^*, v^* \in X^*$ and assuming that there are $u, v \in X$ such that $\|u\|, \|v\| \leq 1$ and $1 = u^*(u) = v^*(v)$, we will prove that

$$\frac{1}{4} \leq \sup\left\{|u^*(x)v^*(x)| : x \in \text{co}\ \{u, \pm v\}\right\}.$$

Actually, for arbitrary nonzero functionals u^*, v^* our requirement will be satisfied by the functionals $\dfrac{\rho u^*}{\|u^*\|}, \dfrac{\rho v^*}{\|v^*\|}$ with $\rho > 1$, so we will get

$$\|u^*\|\|v^*\| \leq 4\rho^2 \sup\{|u^*(x)||v^*(x)| : x \in X, \|x\| \leq 1\}$$

and the desired inequality will follow letting $\rho \to 1$.

Let us write $\alpha = u^*(v)$ and $\beta = v^*(u)$. If $|\alpha| \geq \frac{1}{4}$ or $|\beta| \geq \frac{1}{4}$, the desired inequality holds, since

$$\max\{|u^*(v)v^*(v)|, |u^*(u)v^*(u)|\} = \max\{|\alpha|, |\beta|\} \geq \frac{1}{4}.$$

So let us assume $|\alpha| < \frac{1}{4}, |\beta| < \frac{1}{4}$ and note that $2\alpha\beta \leq |\alpha + \beta|$.

In case that $2\alpha\beta \leq \alpha + \beta$, then we write

$$t = \frac{1-\alpha}{2-\alpha-\beta} \quad \text{and} \quad x = tu + (1-t)v.$$

Since $|\alpha|, |\beta| < \frac{1}{4}$, then t belongs to the interval $[0,1]$ and the element x, which is in the unit ball of X satisfies

$$\begin{aligned} u^*(x)v^*(x) &= \frac{\Big(1-\alpha+(1-\beta)\alpha\Big)\Big((1-\alpha)\beta+1-\beta\Big)}{(2-\alpha-\beta)^2} \\ &= \frac{(1-\alpha\beta)^2}{(2-\alpha-\beta)^2}. \end{aligned}$$

We are assuming $2\alpha\beta \leq \alpha + \beta$, so $(1-\alpha\beta)/(2-\alpha-\beta) \geq \frac{1}{2}$, which implies $u^*(x)v^*(x) \geq \frac{1}{4}$.

On the other hand, if we assume $2\alpha\beta \leq -\alpha - \beta$, then we take

$$t = \frac{1+\alpha}{2+\alpha+\beta} \quad \text{and} \quad x = tu - (1-t)v;$$

again x is in the unit ball of X, since it is in the convex hull of u and $-v$ and satisfies

$$\begin{aligned} |u^*(x)v^*(x)| &= \frac{\Big|\Big(1+\alpha-(1+\beta)\alpha\Big)\Big((1+\alpha)\beta-1-\beta\Big)\Big|}{(2+\alpha+\beta)^2} \\ &= \frac{(1-\alpha\beta)^2}{(2+\alpha+\beta)^2}. \end{aligned}$$

The inequality $2\alpha\beta \leq -(\alpha+\beta)$ gives $(1-\alpha\beta)/(2+\alpha+\beta) \geq \frac{1}{2}$, so it also happens in this case $|u^*(x)v^*(x)| \geq \frac{1}{4}$.

Finally, to show that 4 is the best possible constant in our inequality it is enough to take $X = \mathbb{R}^2$ with norm given by

$$\|(a,b)\| = |a| + |b| \qquad (a, b \in \mathbb{R})$$

and the coordinate functionals

$$u^*(a,b) = a, \quad v^*(a,b) = b \qquad (a, b \in \mathbb{R}).$$

Then $\|u^*\| = \|v^*\| = 1$ and

$$\sup\{|u^*(x)||v^*(x)| : x \in X, \|x\| \leq 1\} = \sup\{|ab| : a, b \in \mathbb{R}, |a| + |b| \leq 1\} = \frac{1}{4}. \blacksquare$$

To finish with, let us note that, according to a result due to C. Benítez and Y. Sarantopoulos [2], if the inequality appearing in the Lemma is satisfied for a normed space X with constant $K = 2$, then X is prehilbertian. For other results in the same line, we refer the interested reader to [1], where the authors show that a real normed space is prehilbertian if, and only if, the norm of every bounded and symmetric m-linear form on it is equal to the norm of its associated polynomial. Also C. Benítez kindly informed us that the corresponding version of the inequality appearing in $(*)$ for m functionals (and so for m operators because of the appropriate and inmediate version of the Lemma) on a complex space was shown in [2] with constant m^m.

Acknowledgments

The author would like to express her gratitude to Miguel Cabrera García and Angel Rodríguez Palacios, who suggested the question dealt with and encouraged her to write this note. Thanks also to Rafael Payá Albert for informing her about some of the results appearing in [2].

References

[1] Benítez, C., Sarantopoulos, Y., Characterization of real inner product spaces by means of symmetric bilinear forms, *J. Math. Anal. Appl.*, **180** (1993), 207–220.

[2] Benítez, C., Sarantopoulos, Y., On the norm of the product of continuous linear functionals in normed spaces, preprint, Universidad de Extremadura, 1995.

[3] Cabrera, M., Rodríguez, A., Nondegenerately ultraprime Jordan-Banach algebras: A zel'manovian treatment, *Proc. London Math. Soc.*, **69** (1994), 576–604.