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Polynomials and Geometry of Banach Spaces†

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In this paper we survey a large part of the results on polynomials on Banach spaces that have been obtained in recent years. We mainly look at how the polynomials behave in connection with certain geometric properties of the spaces. The paper is divided in 8 sections and includes a rather long list of references.

Section 1 is an introduction to polynomials on Banach spaces. It includes the polarization formulae and some important properties of superreflexive spaces. Section 2 is devoted to Nachbin's theorem on approximation of differentiable functions, which provides a motivation to introduce the weakly continuous polynomials, as we do in section 3. This section contains the main results on polynomials whose restrictions to bounded subsets are weakly continuous, highlighting the relationship between this continuity property and the containment of ℓ_1.

In section 4 we study the action of polynomials and multilinear mappings on sequences with upper and lower ℓ_p-estimates. This is applied to obtain conditions for compactness and weak sequential continuity of polynomials. Section 5 is devoted to the Dunford-Pettis property and the polynomial Dunford-Pettis property. New conditions are given in this setting for weak sequential continuity of polynomials. Section 6 focuses on the study of certain properties

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defined in terms of polynomials. We first look at the polynomially Schur and polynomially Dunford-Pettis spaces, and investigate the relationship between both classes. We consider the polynomial Grothendieck and Pełczyński's (V) properties, and their relation to polynomial reflexivity, as well as Q-reflexivity.

In section 7 we introduce the so called properties (P) and (RP) of Banach spaces. We show the interest of these properties in order to give estimates by polynomials, and we mention a few open problems. Section 8 deals with what is called polynomial continuity. We characterize the Banach spaces with polynomially continuous norm and give some results on polynomial continuity on ℓ_1 and $C(K)$ spaces.

The paper treats the various subjects in a rather complete form. Thanks to the choice of references, we hope that reading the paper in detail will be a natural and ordered way to introduce oneself into this interesting, current subject in the field of Functional Analysis and its applications.

1. Introduction to polynomials

There are three approaches to the study of polynomials: by considering restrictions to finite dimensional spaces, by using multilinear mappings, and by means of tensor products. We confine ourselves to the second one.

In 1931–1932, Michal, a student of Fréchet, gave a series of lectures at the California Institute of Technology, in which he outlined the relationship between symmetric, m-linear forms and homogeneous polynomials. Further work on the definition of polynomials between Banach spaces was carried out by Michal and his students Clifford [71], Martin [72, 69], Highberg [50] and Taylor [87]. For modern exposition on symmetric, m-linear forms and homogeneous polynomials, standard references are [28] and [73] (see also [16]).

Throughout, X and Y will stand for Banach spaces over the field $\mathbb{K}$ (real $\mathbb{R}$ or complex $\mathbb{C}$), X^* for the dual of X, and B_X for its closed unit ball. We denote by $\mathcal{L}(^mX, Y)$ the space of all continuous, m-linear mappings from $X^m := X \times \overset{(m)}{\cdots} \times X$ into Y. If $m = 1$, we just have the continuous, linear mappings (often referred to simply as *operators*), and the corresponding space is denoted by $\mathcal{L}(X, Y)$. We define $\mathcal{L}(^0X, Y)$ to be the set of all constant mappings from X into Y, and this space is identified with Y in a natural manner. When $F = \mathbb{K}$, we write $\mathcal{L}(^mX)$ in place of $\mathcal{L}(^mX, K)$. The set of natural numbers is denoted by $\mathbb{N}$ (from the context, it will be clear if it includes 0 or not).

To define continuous, m-homogeneous polynomials, we use the natural embedding, called the diagonal mapping $\Delta_m : X \to X^m$, given by

$$\Delta_m(x) := x^m := (x, \overset{(m)}{\dots}, x).$$

DEFINITION 1.1. A mapping $P : X \to Y$ is a *(continuous) m-homogeneous polynomial* if it may be written in the form $P = L \circ \Delta_m$, where $L \in \mathcal{L}(^mX, Y)$. We say that m is the *degree* of P.

Let $\mathcal{P}(^mX, Y)$ be the vector space of all continuous, m-homogeneous polynomials from X into Y. If $P \in \mathcal{P}(^mX, Y)$, we can find $L \in \mathcal{L}(^mX, Y)$ so that $P(x) = L(x, \dots, x)$. Clearly, $P(\lambda x) = \lambda^m P(x)$, for all $\lambda \in \mathbb{K}$ and $x \in X$. A *continuous polynomial* P from X into Y is any finite sum of continuous, homogeneous polynomials from X into Y. The *degree* of P is the maximum degree of its summands. The space of all continuous polynomials from X into Y is denoted by $\mathcal{P}(X, Y)$.

EXAMPLE. If L is a bilinear form on $\mathbb{K}^n$, $n \in \mathbb{N}$, then there exists an $n \times n$ matrix $A = (a_{ij})$, such that $L(z, w) = zAw^t$, where $z = (z_1, \dots, z_n) \in \mathbb{K}^n$, $w = (w_1, \dots, w_n) \in \mathbb{K}^n$, and w^t is the transpose of w. Thus,

$$L(z, w) = \sum_{i,j=1}^{n} a_{ij} z_i w_j \,.$$

Hence, any $\mathbb{K}$-valued continuous, two-homogeneous polynomial P on $\mathbb{K}^n$ has the familiar form

$$P(z) = L(z, z) = \sum_{i,j=1}^{n} a_{ij} z_i z_j \,.$$

If we replace A by its associated symmetric matrix $B = (A + A^t)/2$, then $B = (b_{ij})$ with $b_{ij} = (a_{ij} + a_{ji})/2$, and $zAz^t = zBz^t$, for every $z \in \mathbb{K}^n$. So, A and B define the same continuous, two-homogeneous polynomial. The previous example, which is typical of the general situation, shows that different multilinear forms may define the same polynomial. To obtain one-to-one correspondence, we must restrict ourselves to symmetric, multilinear mappings. We denote by $\mathcal{L}_s(^mX, Y)$ the linear subspace of $\mathcal{L}(^mX, Y)$ consisting of all continuous symmetric, m-linear mappings. For each $L \in \mathcal{L}(^mX, Y)$, we can define a symmetric, m-linear mapping, in a canonical fashion, given by

$$S(L)(x_1, \dots, x_m) := \frac{1}{m!} \sum_{\sigma \in S_m} L\left(x_{\sigma(1)}, \dots, x_{\sigma(m)}\right) \,,$$

where S_m is the set of all permutations of the first m natural numbers. We call S the *symmetrization operator.* As a consequence of the polarization formula, we shall show that the mapping $\Lambda : \mathcal{L}_s(^mX, Y) \to \mathcal{P}(^mX, Y)$, given by $\Lambda(L) = \hat{L} := L \circ \Delta_m$, is a vector space isomorphism. Therefore, to each $P \in \mathcal{P}(^mX, Y)$ we can associate a unique symmetric, m-linear mapping $\check{P} \in \mathcal{L}_s(^mX, Y)$ so that $P(x) = \check{P}(x, \dots, x) = \check{P}(x^m)$ for all $x \in X$, and a linear, bounded operator $T_P : X \to \mathcal{L}_s(^{m-1}X, Y)$ given by $T_P(x)(x_1, \dots, x_{m-1}) = \check{P}(x, x_1, \dots, x_{m-1})$.

The n th Rademacher function r_n is defined on $[0, 1]$ by

$$r_n(t) := \operatorname{sign} \sin 2^n \pi t.$$

The following polarization formulae allow us to recover the values of a symmetric m-linear mapping from the values of the associated m-homogeneous polynomial.

Polarization formulae

If $P \in \mathcal{P}(^mX, Y)$ and $L \in \mathcal{L}_s(^mX, Y)$ are such that $\hat{L} = P$, then

$$\text{(a)} \qquad L(x_1, \dots, x_m) = \frac{1}{m!} \int_0^1 r_1(t) \cdot \ldots \cdot r_m(t)\, P\left(\sum_{i=1}^m r_i(t) x_i\right) dt$$

$$\text{(b)} \qquad L(x_1, \dots, x_m) = \frac{1}{2^m m!} \sum_{\epsilon_i = \pm 1} \epsilon_1 \cdot \ldots \cdot \epsilon_m\, P\left(\sum_{i=1}^m \epsilon_i x_i\right)$$

(see [73, Theorem 1.10] and [10, p. 21]).

Endowed with the norm

$$\|P\| := \sup_{x \neq 0} \frac{\|P(x)\|}{\|x\|^m} = \sup\left\{\|P(x)\| : \|x\| \leq 1\right\},$$

the space $\mathcal{P}(^mX, Y)$ is a Banach space.

PROPOSITION 1.1. *For every $L \in \mathcal{L}_s(^mX, Y)$, we have*

$$\|\hat{L}\| \leq \|L\| \leq \frac{m^m}{m!} \|\hat{L}\|.$$

This was proved by Martin [69], using the polarization formulae. Thus, $\mathcal{L}_s(^mX, Y)$ and $\mathcal{P}(^mX, Y)$ are isomorphic Banach spaces. The constant $m^m/m!$ is best possible, as shown by the following example due to Nachbin [74, §3, Remark 1].

EXAMPLE. Let $E = \ell_1$. If $x = (x_n) \in \ell_1$, consider the continuous, m-homogeneous polynomial $\hat{L}(x) = P(x) = x_1 \cdot x_2 \cdot \ldots \cdot x_m$. If (e_i) is the unit vector basis of ℓ_1, then $L(e_1, \ldots, e_m) = 1/m!$. So, $\|L\| \geq 1/m!$. On the other hand, $|\hat{L}(x)| \leq 1/m^m$, for $\|x\| = \sum_{i=1}^{\infty} |x_i| \leq 1$. If $x = (1/m, \overset{(m)}{\ldots}, 1/m, 0, 0, \ldots)$, then $\hat{L}(x) = 1/m^m$. Thus

$$\|\hat{L}\| = \frac{1}{m^m}, \text{ and so } \|L\| = \frac{m^m}{m!}\|\hat{L}\|.$$

In this case, we say that L is an *extremal* continuous symmetric, m-linear form on ℓ_1. It is interesting to point out that if there exists an extremal $L \in \mathcal{L}_s({}^m X)$, then ℓ_1^m is finitely representable in X [84]. Recall that Y is said to be *finitely representable in* X [53] (we shall write Y f.r. X), if for every $\epsilon > 0$ and every finite dimensional subspace Y_0 of Y, there are a finite dimensional subspace X_0 of X, and a surjective isomorphism $T : Y_0 \to X_0$ such that

$$\|T\| \cdot \|T^{-1}\| \leq 1 + \epsilon.$$

The following remarkable theorem is due to Pisier:

THEOREM 1.1. ([92, Theorem III.C.16]) *Let X be an infinite dimensional Banach space. Then the following conditions are equivalent:*

(a) X does not have type p, for any $p > 1$.

(b) For every $\epsilon > 0$ and every $m \in \mathbb{N}$, there exist norm-one vectors $x_1, \ldots, x_m \in X$ such that

$$\min_{\epsilon_i = \pm 1} \left\| \sum_{i=1}^{m} \epsilon_i x_i \right\| \geq m - \epsilon.$$

(c) ℓ_1 is finitely representable in X.

Recall that a Banach space X is *of type p*, $1 \leq p \leq 2$ (respectively, *of cotype q*, $2 \leq q \leq \infty$) if there exists a constant $C > 0$ such that for any finite family $x_1, \ldots, x_n \in X$:

$$\frac{1}{2^n} \sum_{\epsilon_i = \pm 1} \left\| \sum_{i=1}^{n} \epsilon_i x_i \right\| \leq C \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p},$$

respectively,

$$\left(\sum_{i=1}^{n} \|x_i\|^q \right)^{1/q} \leq C \frac{1}{2^n} \sum_{\epsilon_i = \pm 1} \left\| \sum_{i=1}^{n} \epsilon_i x_i \right\|.$$

It is easy to show that ℓ_1 is finitely representable in the reflexive space

$$\left(\bigoplus_{n=1}^{\infty} \ell_1^{(n)}\right)_2$$

of all sequences $x = (x_n)$, with $x_n \in \ell_1^{(n)}$ and $\sum \|x_n\|_1^2 < \infty$, endowed with the norm $\|x\| := \left(\sum \|x_n\|_1^2\right)^{1/2}$. Therefore, reflexivity is not preserved under finite representability. As for uniform convexity, however, we get

PROPOSITION 1.2. ([12, 4.I §1, Proposition 1]) *If X is uniformly convex, and Y f.r. X, then Y is uniformly convex.*

Recall that X is *uniformly convex* if $\delta(\epsilon) > 0$ for all $\epsilon > 0$, where the function

$$\delta(\epsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1,\ \|x - y\| \geq \epsilon\right\}$$

is called the *modulus of convexity* of X.

A space X is *superreflexive* if

$$Y \text{ f.r. } X \Longrightarrow Y \text{ is reflexive.}$$

Hence, the above mentioned space $\left(\oplus_n \ell_1^{(n)}\right)_2$ is reflexive, but not superreflexive. Since every uniformly convex space is reflexive [64, Proposition 1.e.3], Proposition 1.2 shows that every uniformly convex space is superreflexive. Using geometric characterizations of superreflexivity, James proved that X is superreflexive if and only if X^* is superreflexive [52]. Since X^* is uniformly convex when X is uniformly smooth [64, Proposition 1.e.2], it follows that every uniformly smooth space is superreflexive. Recall that a Banach space X is *uniformly smooth* if

$$\lim_{\tau \to 0} \frac{\rho(\tau)}{\tau} = 0\,,$$

where

$$\rho(\tau) := \sup\left\{\frac{1}{2}\left(\|x+y\| + \|x-y\|\right) - 1 : \|x\| \leq 1, \|y\| \leq \tau\right\}$$

is called *modulus of smoothnes* of X.

The following result of James' has been very useful in the theory of polynomials:

THEOREM 1.2. ([53]) *Let X be superreflexive. Then, for every $C > 1$ and $K \geq 1$, there is a number $p \in (1, \infty)$ such that, for every normalized basic sequence (x_n), with basis constant K, and for every finite sequence (a_i) of scalars, we have*

$$\left\| \sum_i a_i x_i \right\| \leq C \left(\sum_i |a_i|^p \right)^{1/p} .$$

In particular, if (e_i) is the unit vector basis of ℓ_p, there is an operator $T : \ell_p \to X$, so that $Te_i = x_i$.

We pointed out that uniformly convex and uniformly smooth spaces are superreflexive. It was proved by Enflo [32] that the converse is also true, in the sense that every superreflexive space can be given an equivalent norm which is uniformly convex, and also an equivalent, uniformly smooth norm. This result was refined by Pisier, who gave an estimate for the moduli of convexity and smoothness:

THEOREM 1.3. ([78]) *Let X be superreflexive. Then:*

*(a) There are $p > 2$, $C > 0$ and an equivalent norm on X whose modulus of convexity $\delta(\epsilon)$ satisfies $\delta(\epsilon) \geq C\epsilon^p$ for all $\epsilon > 0$ (we say that X is p-*convex*).*

*(b) There are $q \in (1, 2)$, $C' > 0$ and an equivalent norm on X whose modulus of smoothness $\rho(\tau)$ satisfies $\rho(\tau) \leq C'\tau^q$ (we say that X is q-*smooth*).*

We shall be interested in the subspace $\mathcal{P}_f(^mX, Y)$ of $\mathcal{P}(^mX, Y)$ generated by the collection of functions

$$\phi^m \otimes y = \phi^m \cdot y \quad (m \in \mathbb{N},\ \phi \in X^*,\ y \in Y),$$

where $(\phi^n \otimes y)(x) := \phi^n(x) \cdot y$, for each $x \in X$. Let

$$\mathcal{P}_f(X, Y) = \sum_{m=0}^{\infty} \mathcal{P}_f(^mX, Y)$$

be the space of all continuous polynomials of *finite type* from X into Y. The completion of $\mathcal{P}_f(^mX, Y)$ with respect to the norm induced by $\mathcal{P}(^mX, Y)$ is denoted by $\mathcal{P}_c(^mX, Y)$, and is in general strictly contained in $\mathcal{P}(^mX, Y)$. Let K be a compact Hausdorff space, and $C(K)$ the space of all scalar valued, continuous functions on K. If K is *dispersed* (every closed subset of K contains an isolated point), and $X = C(K)$ with the supremum norm, then for every $m \in \mathbb{N}$, $\mathcal{P}(^mX) = \mathcal{P}_c(^mX)$ [3, p. 215].

The following Banach-Steinhaus type theorem holds:

PROPOSITION 1.3. ([70]) *Let X, Y be Banach spaces, and let (P_n) be a sequence in $\mathcal{P}(^mX,Y)$. If (P_n) converges pointwise to a mapping $P : X \to Y$, then $P \in \mathcal{P}(^mX,Y)$.*

A polynomial is *(weakly) compact* if it takes the unit ball into a relatively (weakly) compact subset.

2. NACHBIN'S THEOREM

In this section, X and Y will be real Banach spaces. Given $m \in \mathbb{N} \cup \{+\infty\}$, we endow the space $C^m(X,Y)$ of m-times continuously Fréchet differentiable, Y-valued functions on X with the compact-open topology of order m, i.e., the topology of uniform convergence of the functions and their derivatives of order $\leq m$ on the compact subsets of X. We denote this topology by τ_c^m. Recall that, for $j \leq m$, the jth derivative of $f \in C^m(X,Y)$ is a mapping $d^jf : X \to \mathcal{P}(^jX,Y)$. We denote $df := d^1f$.

In the theory of polynomial approximation of differentiable functions there are essentially two basic results for finite dimensional spaces: Weierstrass' and Nachbin's theorems. The first assures that the space $\mathcal{P}(\mathbb{R}^n)$ is τ_c^m-dense in $C^m(\mathbb{R}^n)$. In 1948, Nachbin went to the University of Chicago for a two year visit, at the invitation of Stone. While there, he had the opportunity, in 1949, to present at André Weil's Seminar a recent article by Whitney [91].

Pressed by Segal's question in 1949, Nachbin studied the noteworthy case of dense subalgebras, to obtain:

THEOREM 2.1. ([75]) *Let X be a real, Hausdorff C^∞-manifold of finite dimension n, and $G \subset C^m(X)$, with $m \in \mathbb{N} \cup \{+\infty\}$. Then the algebra generated by G is τ_c^m-dense in $C^m(X)$ if and only if the following conditions hold:*

(a) G separates points.

(b) G does not vanish at any point.

(c) For every $x \in X$ and $v \in T_x(X)$, the tangent space at x, $v \neq 0$, there exists $g \in G$ such that $dg(x)(v) \neq 0$.

In 1975, trying to extend Nachbin's theorem to infinite dimensional spaces, Lesmes [60] and Llavona [66] found the following counterexample which shows that the space $\mathcal{P}_f(X)$, a basic example of subalgebra satisfying the conditions of Theorem 2.1, is not dense in $C^m(X)$, when $m \geq 2$ and X is an infinite dimensional Hilbert space.

COUNTEREXAMPLE. Let X be a real infinite dimensional Hilbert space. The space $\mathcal{P}_f(X)$ is not τ_c^2-dense in $C^2(X)$. The point is that if $f : X \to \mathbb{R}$ is defined by $f(x) = \|x\|^2$, then $d^2f(x) = 2\mathrm{Id}$, where Id is the identity map on X. Since for each $P \in \mathcal{P}_f(X)$, $d^2P(x) \in \mathcal{L}(X, H)$, where H is a finite dimensional subspace of X, it follows that $d^2f(x) \notin \mathcal{P}_c(^2X)$. Therefore, in order to get polynomial approximation for the τ_c^m-topology, it seems logical to consider functions $f \in C^m(X,Y)$ so that $d^j f(x) \in \mathcal{P}_c(^jX, Y)$, for every $x \in X$ and $j \leq m$.

For different extensions of Theorem 2.1 to infinite dimensional spaces, we refer the reader to Llavona's book [67].

With the study of the space $\mathcal{P}_c(^mX, Y)$, a particular, important class of polynomials is introduced in this theory, the polynomials whose restrictions to bounded subsets are weakly (respectively, weakly uniformly) continuous. We concentrate our attention on this class of polynomials in the next section.

3. WEAKLY CONTINUOUS POLYNOMIALS

If $A \subseteq X$, a function $f : A \to Y$ is said to be *weakly continuous* if for each $x \in A$ and $\epsilon > 0$, there are $\phi_1, \ldots, \phi_n \in X^*$ and $\delta > 0$ such that $\|f(x) - f(y)\| < \epsilon$ whenever $|\phi_i(x - y)| < \delta$ for $y \in A$ and $i = 1, \ldots, n$. We denote by $\mathcal{P}_{wb}(^mX, Y)$ the space of all m-homogeneous polynomials from X into Y whose restrictions to bounded subsets of X are weakly continuous. A function $f : A \to Y$ is said to be *weakly uniformly continuous* if for each $\epsilon > 0$ there are $\phi_1, \ldots, \phi_n \in X^*$ and $\delta > 0$ such that $\|f(x) - f(y)\| < \epsilon$, whenever $|\phi_i(x - y)| < \delta$ for $x, y \in A$ and $i = 1, \ldots, n$. We denote by $\mathcal{P}_{wbu}(^mX, Y)$ the space of m-homogeneous polynomials whose restrictions to bounded subsets of X are weakly uniformly continuous.

THEOREM 3.1. ([11, Proposition 2.7]) *If X^* has the approximation property, then $\mathcal{P}_{wbu}(^mX, Y) = \mathcal{P}_c(^mX, Y)$ for all m.*

An introduction to the approximation property may be seen in [63, 1.e].

It is easy to show that if a (non necessarily linear) mapping $f : X \to Y$ is weakly uniformly continuous on bounded subsets of X, then f takes bounded sets into relatively compact sets [11, Lemma 2.2]. Valdivia [89] showed that a Banach space X is reflexive if and only if every weakly continuous, scalar valued function on X is bounded on bounded sets. Therefore, X is reflexive if and only if each mapping from X into any Banach space which is weakly continuous on bounded sets is in fact weakly uniformly continuous on bounded sets. However, as for polynomials, we have the following remarkable theorem:

THEOREM 3.2. ([9, Theorem 2.9]) *Let $P \in \mathcal{P}(^mX, Y)$ and let the associated linear mapping $T_P : X \to \mathcal{L}_s(^{m-1}X, Y)$ be given. Then $P \in \mathcal{P}_{wb}(^mX, Y)$ if and only if T_P is compact. Consequently, $\mathcal{P}_{wbu}(^mX, Y) = \mathcal{P}_{wb}(^mX, Y)$.*

We denote by $\mathcal{P}_{wsc}(^mX, Y)$ the space of those $P \in \mathcal{P}(^mX, Y)$ which are *weakly sequentially continuous* (w.s.c., for short), i.e., such that for every sequence $(x_n) \subset X$ weakly converging to x, the sequence $(P(x_n))$ converges to $P(x)$ in norm. It is clear that

$$\mathcal{P}_{wbu}(^mX, Y) = \mathcal{P}_{wb}(^mX, Y) \subseteq \mathcal{P}_{wsc}(^mX, Y).$$

EXAMPLE. Let $P(x) := \sum_{n=1}^{\infty} x_n^2$, acting either on ℓ_1 or on ℓ_2. Then $P \notin \mathcal{P}_{wsc}(^2\ell_2)$, since the unit vector basis (e_i) in ℓ_2 is weakly null, but $P(e_n) = 1$ for all n. On the other side, $P \in \mathcal{P}_{wsc}(^2\ell_1)$, since weak and norm convergence of sequences coincide in ℓ_1. However, $P \notin \mathcal{P}_{wbu}(^2\ell_1)$.

The following result gives a full description of the relationship between $\mathcal{P}_{wb}(^mX, Y)$ and $\mathcal{P}_{wsc}(^mX, Y)$. The equivalence (a)⇔(b) was proved in [38], and the other statements in [48].

THEOREM 3.3. *The following assertions are equivalent:*

(a) *X contains no copy of ℓ_1.*

(b) *For every Y, we have $\mathcal{P}_{wb}(^mX, Y) = \mathcal{P}_{wsc}(^mX, Y)$ for all $m \in \mathbb{N}$.*

(c) *For some Y and some $m \in \mathbb{N}$, we have $\mathcal{P}_{wb}(^mX, Y) = \mathcal{P}_{wsc}(^mX, Y)$.*

(d) *For some $m \in \mathbb{N}$, we have $\mathcal{P}_{wb}(^mX) = \mathcal{P}_{wsc}(^mX)$.*

Examples of spaces such that $\mathcal{P}(^mX) = \mathcal{P}_{wb}(^mX)$ for all m are:

(a) The original Tsirelson space T^* [1, 88].

(b) The quasi-reflexive Tsirelson-James space T_J^* [7, 68] (recall that a space is *quasi-reflexive* if it has finite codimension in its bidual).

(c) In general, every space with property (S_p) for all $p > 1$ (see definition below), not containing ℓ_1 [45], such as Leung's space [61].

Recall that a sequence $(x_n) \subset X$ is called *seminormalized* if there are constants $K, k > 0$ such that $k \le \|x_n\| \le K$ for all n. If $1 \le p, q \le \infty$, we say that a sequence (x_n) has an *upper p-estimate* (respectively, a *lower* *q-estimate*) if there is a constant $C > 0$ such that for every $n \in \mathbb{N}$ and every $a_1, \dots, a_n \in \mathbb{K}$, we have

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| \le C \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p},$$

respectively,

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| \ge C \left(\sum_{i=1}^{n} |a_i|^q \right)^{1/q}.$$

where, if $p = \infty$ (resp. $q = \infty$), the right hand side has to be replaced by $\sup_i |a_i|$. In particular, a normalized, basic sequence has an upper ∞-estimate if and only if it is equivalent to the unit vector basis of c_0.

We now introduce other related sequential properties, which will be used in the next section. Following [58], we say that X has *property* (S_p) $(1 \le p \le \infty)$ if every weakly null seminormalized, basic sequence in X has a subsequence with an upper p-estimate. Similarly, X has *property* (T_q) $(1 < q \le \infty)$ if every weakly null seminormalized, basic sequence has a subsequence with a lower q-estimate.

A Banach space X has the *Schur property* (we shall say that X *is Schur,* for short) if every weakly null sequence in X is norm null. The space ℓ_1 is Schur. Since a Schur space has properties (S_p) and (T_q) for all $1 \le p \le \infty$ and $1 < q \le \infty$, we say that X has *property* (T_1) if and only if X is Schur. Every Banach space has properties (S_1) and (T_∞).

A sequence $(x_n) \subset X$ is said to be *weakly p-summable* if for every $\phi \in X^*$, we have that $(\phi(x_n)) \in \ell_p$ (or $(\phi(x_n)) \in c_0$ if $p = \infty$). It is worth noting that a sequence has an upper p-estimate if and only if it is weakly q-summable, where $p^{-1} + q^{-1} = 1$.

The sequence $(x_n) \subset X$ is *weakly p-convergent* if there is $x \in X$ such that $(x_n - x)_{n=1}^{\infty}$ is weakly p-summable. Following [21], we say that X is in the *class* W_p if every bounded sequence in X has a weakly p-convergent subsequence. Clearly,

$$W_1 \subseteq W_2 \subseteq \cdots \subseteq W_\infty \equiv \{\text{Reflexive spaces}\}.$$

Using James' Theorem 1.2 and the Bessaga-Pełczyński selection principle [14], we obtain (see [21]):

$$X \text{ superreflexive} \implies X \in W_p \text{ for some } p > 1.$$

4. Polynomials and ℓ_p-estimates of sequences

One of the earliest results connecting polynomials (or multilinear mappings) with properties of weak convergence of sequences is due to Littlewood [65], who showed that every bounded, bilinear form on c_0 is approximable by bilinear forms of finite type, that is, finite linear combinations of products of functionals on c_0. This result was extended by Pitt [79], who proved that every bounded, bilinear form on $\ell_p \times \ell_q$ is approximable by bilinear forms of finite type, whenever $p^{-1} + q^{-1} < 1$. Taking into account that bilinear forms on $X \times Y$ correspond precisely to linear operators from X into Y^*, we obtain that every linear operator from c_0 into ℓ_1 is compact, and so is every linear operator from ℓ_p into ℓ_q, for $p > q$. Of course, we also obtain that every 2-homogeneous, scalar valued polynomial on c_0 (resp., on ℓ_p, $p > 2$) is weakly sequentially continuous. These results were extended by Pełczyński [76] who proved that every scalar valued polynomial on c_0 is weakly sequentially continuous (see also [17]), and every N-homogeneous polynomial from ℓ_p into ℓ_q is compact whenever $p > Nq$.

Our aim in this section is to provide conditions on X and Y so that every polynomial of a given degree from X into Y be compact. This will be achieved by studying the action of polynomials and multilinear mappings on sequences with either upper or lower ℓ_p-estimates. By Theorem 3.3, if X contains no copy of ℓ_1, then every w.s.c. polynomial from X into Y is compact, for every Y. On the other hand, if X does contain a copy of ℓ_1 and Y is infinite dimensional, then there exists a non-compact N-homogeneous polynomial from X into Y, for each $N \geq 2$ [44, II.2]. Therefore, we shall mainly be concerned with weak sequential continuity of polynomials.

A sequence $(x_n) \subset X$ is said to be *p-Banach-Saks* $(1 < p < \infty)$ if there is $C > 0$ such that for all $n \in \mathbb{N}$, we have

$$\left\| \sum_{k=1}^{n} x_k \right\| \leq C n^{1/p}.$$

We say that (x_n) is *hereditarily p-Banach-Saks* if there is $C > 0$ such that any subsequence of (x_n) is p-Banach-Saks with constant C. A formal series $\sum_{i=1}^{\infty} x_i$ in X is *weakly unconditionally Cauchy* (w.u.C., for short) if for every $\phi \in X^*$, we have that $\sum_{i=1}^{\infty} |\phi(x_i)| < \infty$. We start with a permanence result.

THEOREM 4.1. *Let P be an N-homogeneous polynomial from X into Y, $(x_j) \subset X$ a sequence, and $p > N$. Then:*

(a) ([10]) *If* (x_j) *is hereditarily* p*-Banach-Saks, then* $(P(x_j))$ *is hereditarily* p/N*-Banach-Saks.*

(b) ([45]) *If* (x_j) *has an upper* p*-estimate, then* $(P(x_j))$ *has an upper* p/N*-estimate.*

(c) ([42]) *If the series* $\sum x_j$ *is w.u.C., then* $\sum P(x_j)$ *is w.u.C. as well (this is the case* $p = \infty$*).*

A basic tool in the proof of this theorem is the sequence of generalized Rademacher functions, introduced in [8], and defined as follows:

Fix $2 \leq k \in \mathbb{N}$, and let $\alpha_1 = 1, \alpha_2, \dots, \alpha_k$ denote the kth roots of unity.

Let $s_1 : [0,1] \to \mathbb{C}$ be the step function taking the value α_j on the interval $((j-1)/k, j/k)$ for $j = 1, \dots, k$.

Then, assuming that s_{n-1} has been defined, define s_n as follows. Fix any of the k^{n-1} subintervals I of $[0,1]$ used in the definition of s_{n-1}. Divide I into k equal intervals $I_1, \dots, I_k$, and set $s_n(t) := \alpha_j$ if $t \in I_j$.

These functions are orthogonal [8, Lemma 1.2] in the sense that, for any choice of integers $i_1, \dots, i_k$; $k \geq 2$, we have

$$\int_0^1 s_{i_1}(t) \cdot \ldots \cdot s_{i_k}(t)\, dt = \begin{cases} 1, & \text{if } i_1 = \cdots = i_k; \\ 0, & \text{otherwise.} \end{cases}$$

As an example, we shall give the proof of Theorem 4.1(c), which needs the following previous lemma:

LEMMA 4.1. ([42]) *Let* X, Y *be complex Banach spaces. Given a polynomial* $P \in \mathcal{P}(^kX; Y)$*, we have that, for every* $x_1, ..., x_n \in X$,

$$\sup_{|\epsilon_j| \leq 1} \left\| \sum_{j=1}^n \epsilon_j P x_j \right\| \leq \sup_{|\nu_j| \leq 1} \left\| P\left(\sum_{j=1}^n \nu_j x_j \right) \right\| .$$

Proof. Observe that both suprema are attained for some $|\epsilon_j| = |\nu_j| = 1$ $(j = 1, \dots, n)$.

For any $x_1, ..., x_n \in X$ and any complex numbers ϵ_j with $|\epsilon_j| = 1$, we can find $\psi \in Y^*$, $\|\psi\| = 1$, such that

$$\left\| \sum_{j=1}^n \epsilon_j P x_j \right\| = \psi\left(\sum_{j=1}^n \epsilon_j P x_j \right) .$$

Then, choosing complex numbers δ_j such that $\delta_j^k = \epsilon_j$, we obtain

$$\begin{aligned}
\left\|\sum_{j=1}^{n} \epsilon_j P x_j\right\| &= \psi\left(\sum_{j=1}^{n} P(\delta_j x_j)\right) \\
&= \int_0^1 \left(\sum_{j_1,\dots,j_k=1}^{n} s_{j_1}(t)\cdot\ldots\cdot s_{j_k}(t)\psi\circ\hat{P}(\delta_{j_1}x_{j_1},\dots,\delta_{j_k}x_{j_k})\right) dt \\
&= \int_0^1 \psi\circ\hat{P}\left(\sum_{j_1=1}^{n}\delta_{j_1}s_{j_1}(t)x_{j_1},\dots,\sum_{j_k=1}^{n}\delta_{j_k}s_{j_k}(t)x_{j_k}\right) dt \\
&= \int_0^1 \psi\circ P\left(\sum_{j=1}^{n}\delta_j s_j(t)x_j\right) dt \\
&\leq \sup_{|\nu_j|=1}\left\|P\left(\sum_{j=1}^{n}\nu_j x_j\right)\right\|,
\end{aligned}$$

and the proof is finished. ∎

We remark that if the spaces under consideration are real, then the right hand side of the inequality has to be multiplied by $(2k)^k/k!$. The proof, using complexifications of the spaces, is standard.

Now, to prove Theorem 4.1(c), it is enough to recall [14, Lemma 2] that a series $\sum x_i$ is w.u.C. if and only if

$$\sup_n \sup_{|\epsilon_i|\leq 1}\left\|\sum_{i=1}^{n}\epsilon_i x_i\right\| < \infty .$$

Theorem 4.1 has a natural counterpart for multilinear mappings:

THEOREM 4.2. *Given Banach spaces $X_1,\dots,X_N,Y$, let $T : X_1\times\cdots\times X_N\to Y$ be a continuous, N-linear mapping. Choose $p_1,\dots,p_N\in(1,+\infty]$ such that*

$$\frac{1}{p_1}+\cdots+\frac{1}{p_N} < 1,$$

and let

$$q := \left(\frac{1}{p_1}+\cdots+\frac{1}{p_N}\right)^{-1}.$$

Consider sequences $\left(x_j^i\right)_{j=1}^{\infty}\subset X_i$, for $1\leq i\leq N$. Then:

(a) [2] If for each $i = 1, \ldots, N$, the sequence $\left(x_j^i\right)_{j=1}^{\infty}$ is hereditarily p_i-Banach-Saks, then the sequence $\left(T\left(x_j^1, \ldots, x_j^N\right)\right)_{j=1}^{\infty}$ is hereditarily q-Banach-Saks.

(b) If for each $i = 1, \ldots, N$, the sequence $\left(x_j^i\right)_{j=1}^{\infty}$ has an upper p_i-estimate, then the sequence $\left(T\left(x_j^1, \ldots, x_j^N\right)\right)_{j=1}^{\infty}$ has an upper q-estimate.

(c) Suppose for each $i = 1, \ldots, N$, the series $\sum_j x_j^i$ is w.u.C. Then the series $\sum_j T\left(x_j^1, \ldots, x_j^N\right)$ is w.u.C. (case $p_1 = \cdots = p_N = \infty$).

As a consequence of Theorem 4.1 or 4.2, we obtain the following result of Aron-Globevnik and Zalduendo:

COROLLARY 4.1. ([8, 93]) *Suppose $X = \ell_p$ $(1 < p < \infty)$ or $X = c_0$, let (e_j) be the usual basis of X, and $P \in \mathcal{P}(^N X)$. Then:*
(a) If $X = \ell_p$ and $N < p$, then $(P(e_j)) \in \ell_{(p/N)^*}$.
(b) If $X = c_0$, then $(P(e_j)) \in \ell_1$.

Next we define the lower and upper indices for a Banach space X, in connection with properties (S_p) and (T_q) introduced at the end of section 3. These indices are the following:

$$\begin{aligned} l(X) : &= \sup\{p \geq 1 : X \text{ has property } S_p\} \in [1, +\infty] \\ u(X) : &= \inf\{q \leq +\infty : X \text{ has property } T_q\} \in [1, +\infty]. \end{aligned}$$

EXAMPLES. [44, 45].
(1) If X is not a Schur space, then $l(X) \leq u(X)$.
(2) If $1 < p < \infty$, then $l(\ell_p) = p = u(\ell_p)$; $l(L_p[0,1]) = \min\{2, p\}$; $u(L_p[0,1]) = \max\{2, p\}$.
(3) $l(\ell_1) = +\infty$; $u(\ell_1) = 1$.
(4) $l(L_1[0,1]) = 1$; $u(L_1[0,1]) = 2$.
(5) $l(c_0) = +\infty = u(c_0)$, and the supremum and infimum are attained.
(6) For the original Tsirelson space T^*, we have $l(T^*) = +\infty = u(T^*)$, and the supremum and infimum are not attained.
(7) For the dual space T of T^*, $l(T) = 1 = u(T)$.
(8) The James space J [51] and its dual J^* satisfy $l(J) = u(J) = 2 = l(J^*) = u(J^*)$.
(9) If M is an Orlicz function satisfying the Δ_2-condition at 0 [63, Definition 4.a.3], then for the Orlicz sequence space ℓ_M we have $l(\ell_M) = \alpha_M$

and $u(\ell_M) = \beta_M$, where α_M and β_M are the lower and upper Boyd indices associated to M, respectively (see [57]).

(10) For $1 < p < 2$,

$$X \text{ is } p\text{-smooth} \implies X \text{ has } (S_p) \implies l(X) \geq p$$
$$X \text{ has type } p \implies X \text{ has } (S_r) \text{ for } 1 < r < p \implies l(X) \geq p\,.$$

For $2 < q < \infty$,

$$X \text{ is } q\text{-convex} \implies X \text{ has } (T_q) \implies u(X) \leq q$$
$$X \text{ has cotype } q \implies X \text{ has } (T_r) \text{ for } r > q \implies u(X) \leq q\,.$$

By using these indices, we obtain the following Theorem, which is essentially a reformulation of some of the results in [76].

THEOREM 4.3. ([45]) *Let X and Y be Banach spaces. Then the following conditions hold:*

(a) If $Nu(Y) < l(X)$, then every N-homogeneous polynomial from (any subspace of) X into Y is w.s.c.

(b) If X has property (S_∞) and Y contains no copy of c_0, then every polynomial from (any subspace of) X into Y is w.s.c.

As above, this Theorem has a natural extension to multilinear mappings (see also [2] for closely related results). Given an N-linear mapping $T : X_1 \times \cdots \times X_N \to Y$, we say that T is *weakly sequentially continuous* (w.s.c.) if, whenever the sequence $\left(x_j^i\right)_{j=1}^{\infty} \subset X_i$ is weakly convergent to x^i, for $1 \leq i \leq N$, we have that $T\left(x_j^1, \ldots, x_j^N\right)$ converges to $T(x^1, \ldots, x^N)$ in norm, as $j \to \infty$.

THEOREM 4.4. ([27]) *Let $X_1, \ldots, X_N, Y$ be Banach spaces. Then the following conditions hold:*

(a) If

$$\frac{1}{l(X_1)} + \cdots + \frac{1}{l(X_N)} < \frac{1}{u(Y)}\,,$$

then every continuous, N-linear mapping from $X_1 \times \cdots \times X_N$ into Y is w.s.c.

(b) If $X_1, \ldots, X_N$ have property S_∞, and Y contains no copy of c_0, then every continuous, N-linear mapping from $X_1 \times \cdots \times X_N$ into Y is w.s.c.

Specializing into ℓ_p-spaces, we have the following extension of Pitt's classical theorem.

THEOREM 4.5. ([2, 27, 39]) *For* $p_1, \dots, p_N, q \in (1, +\infty)$, *the following conditions are equivalent:*

(a)

$$\frac{1}{p_1} + \cdots + \frac{1}{p_N} < \frac{1}{q}.$$

(b) Every continuous, N*-linear mapping from* $\ell_{p_1} \times \cdots \times \ell_{p_N}$ *into* ℓ_q *is compact.*

(c) Every continuous, N*-linear mapping from* $\ell_{p_1} \times \cdots \times \ell_{p_N}$ *into* ℓ_q *is w.s.c.*

(d) The space $\mathcal{L}(\ell_{p_1}, \dots, \ell_{p_N}; \ell_q)$ *of continuous,* N*-linear mappings from* $\ell_{p_1} \times \cdots \times \ell_{p_N}$ *into* ℓ_q *is reflexive.*

As a consequence of Theorem 4.3 we have that, if $N < l(X)$, then every scalar valued, N-homogeneous polynomial on X is w.s.c. (see also [19, Lemma 1] where an analogous result is given). Nevertheless, in the case of scalar valued polynomials, something better can be said. The following result is obtained in [44, 46], refining previous work by Farmer [36].

THEOREM 4.6. *Suppose that no normalized, weakly null sequence in* X *has a lower* q*-estimate. Then, for* $N < q$*, every scalar valued* N*-homogeneous polynomial on* X *is w.s.c.*

The proof of this Theorem makes use in a strong way of the theory of spreading models (see [13] for an extensive treatment). In the proof, the following result is used, which may be of independent interest.

THEOREM 4.7. ([46]) *Let* $(y_n) \subset X$ *be a weakly null sequence which admits a spreading model* E *with unconditional basis* (e_n)*, and let* $P \in \mathcal{P}(^N X)$*. Then there exists a polynomial* $\mathbb{P} \in \mathcal{P}(^N E)$ *and a subsequence* (x_n) *of* (y_n) *such that, for all scalars* $a_1, \dots, a_k$*:*

$$\text{(a)} \qquad \mathbb{P}\left(\sum_{j=1}^{k} a_j e_j\right) = \lim_{n_1 < \cdots < n_k} P\left(\sum_{j=1}^{k} a_j x_{n_j}\right)$$

$$\text{(b)} \qquad \mathbb{P}\left(\sum_{j=1}^{k} a_j e_j\right) = \mathbb{P}\left(\sum_{j=1}^{k} a_j e_{n_j}\right) \quad \text{if } n_1 < \cdots < n_k.$$

Another result on weak sequential continuity of polynomials will be given at the end of the next section on the Dunford-Pettis property.

5. The polynomial Dunford-Pettis property

In 1940, in a now classical Transactions paper [31], M. Dunford and B. Pettis noticed that for a finite measure μ and a separable Banach space X, if $T : L^1(\mu) \to X$ is a weakly compact, linear operator, then T is *completely continuous*, i.e., T takes weakly compact subsets of $L^1(\mu)$ into norm compact subsets of X. In the early 1950's Grothendieck canonized those Banach spaces which share with $L^1(\mu)$ the property that weakly compact operators are completely continuous.

Definition 5.1. ([47]) A Banach space X is said to have the *Dunford-Pettis property* (DPP, for short) if for each Banach space Y, every weakly compact, linear operator $X \to Y$ is completely continuous.

For basic facts about the DPP, the reader is referred to [26], where the following well-known result may be seen:

Theorem 5.1. *The following assertions are equivalent:*

(a) X has the DPP.

(b) For any space Y, every weakly compact operator from X into Y takes weakly convergent sequences into norm convergent sequences.

(c) For any space Y, every weakly compact operator from X into Y takes weak Cauchy sequences into norm convergent sequences.

(d) *For any pair of weakly null sequences $(x_n) \subset X$ and $(\phi_n) \subset X^*$, we have that* $\lim_{n\to\infty} \phi_n(x_n) = 0$.

Easily, the DPP is inherited by complemented subspaces. Considering the identity map, we see that no infinite dimensional, reflexive Banach space can have the DPP. Clearly, every space with the Schur property has the DPP.

Since every weakly null sequence in X is weakly null in X^{**}, we see that X has the DPP whenever X^* has it. The converse is not true. There is essentially one counterexample due to Stegall [85], which we present briefly. Let

$$X = \left(\bigoplus_{n=1}^{\infty} \ell_2^{(n)} \right)_1$$

be the space of all sequences $x = (x_n)$, with $x_n \in \ell_2^{(n)}$ and $\sum \|x_n\|_2 < \infty$, endowed with the norm $\|x\| := \sum \|x_n\|_2$. Then X has the Schur property. However, it is not difficult to see that its dual

$$X^* = \left(\bigoplus_{n=1}^{\infty} \ell_2^{(n)} \right)_\infty$$

contains a complemented copy of ℓ_2, and cannot therefore have the DPP.

The problem of finding additional conditions on X so that if X has the DPP then X^* has it too has been open for a long time. However, it is easy to prove that

THEOREM 5.2. ([35, 77]) *A space X has the DPP and contains no copy of ℓ_1 if and only if X^* has the Schur property (and, therefore, the DPP as well).*

All $C(K)$ and $L_1(\mu)$ spaces have the DPP [47].

In 1979, Ryan showed that the DPP is equivalent to the *polynomial Dunford-Pettis property* (PDPP):

THEOREM 5.3. ([82]) *Suppose X has the DPP, and let Y be any Banach space. If $P : X \to Y$ is a weakly compact polynomial, then $(P(x_n))$ is norm convergent in Y whenever $(x_n) \subset X$ is weak Cauchy.*

In particular, if X has the DPP, then $\mathcal{P}(^mX) = \mathcal{P}_{wsc}(^mX)$ for all m. By Theorem 3.3, we get:

COROLLARY 5.1. *If X has the DPP and contains no copy of ℓ_1, then $\mathcal{P}_{wb}(^mX) = \mathcal{P}(^mX)$ for all m.*

The PDPP was the original motivation to introduce the weak polynomial topology in [20]. We say that a net $(x_\alpha) \subset X$ converges to x in the *weak polynomial topology* (*wp*-topology, for short) if for every $P \in \mathcal{P}(X)$ we have that $P(x_\alpha) \to P(x)$. Recall that X is said to be a Λ-*space* [20] if a sequence $(x_n) \subset X$ is norm null whenever it satisfies $P(x_n) \to 0$ for all $P \in \mathcal{P}(X)$. Trivially, every space with the Schur property is a Λ-space.

Using the fact that every superreflexive space is in W_p for some $p > 1$, it is proved in [54] that every superreflexive space is a Λ-space.

The spaces c_0, $L_1[0,1]$, $L_\infty[0,1]$, ℓ_∞ are not Λ-spaces. In fact, a space with the DPP not satisfying the Schur property is not a Λ-space, since it is easy to prove that

PROPOSITION 5.1. ([20]) *A Banach space has the Schur property if and only if it has the DPP and is a Λ-space.*

By means of techniques similar to those used by Ryan in his proof of Theorem 5.3, the following result may be obtained:

THEOREM 5.4. ([41]) *Suppose X has the DPP, and every operator $X \to Y$ is weakly compact. Then for every $k \in \mathbb{N}$, each continuous, k-linear mapping from X^k into Y is w.s.c.*

COROLLARY 5.2. ([41]) *Suppose X has the DPP, and every operator $X \to Y$ is weakly compact. Then, every polynomial from X into Y is w.s.c.*

In particular, every polynomial from ℓ_∞ into a space containing no copy of ℓ_∞ is w.s.c. However, not every such polynomial is weakly compact [42, Remark 16].

6. OTHER POLYNOMIAL PROPERTIES

In the last few years there has been an increasing interest on Banach space properties defined in terms of polynomials, and also on properties of spaces of polynomials. In this section, we concentrate on some of them. We start with polynomially Schur and polynomially Dunford-Pettis spaces, first studied by Farmer and Johnson in [37]. For a Banach space X, let $\mathcal{P}^{\leq N}(X)$ denote the space of all scalar valued polynomials on X of degree less than or equal to N. A sequence $(x_n) \subset X$ is said to be $\mathcal{P}^N$*-null* (resp. $\mathcal{P}^{\leq N}$*-null*, resp. $\mathcal{P}$*-null*) if $P(x_n) \to 0$ for every $P \in \mathcal{P}(^N X)$ (resp. $\mathcal{P}^{\leq N}(X)$, resp. $\mathcal{P}(X)$).

We say that X is $\mathcal{P}^{\leq N}$-Schur if every $\mathcal{P}^{\leq N}$-null sequence in X is norm null. Obviously, each such space is a Λ-space, and $\mathcal{P}^{\leq 1}$-Schur property is nothing but Schur property. As typical examples, we have that ℓ_p is $\mathcal{P}^{\leq N}$-Schur for $N \geq p$ $(1 \leq p < \infty)$. On the other hand,

$$\left(\bigoplus_{n=1}^{\infty} \ell_{2n} \right)_2$$

is a Λ-space, but it is not $\mathcal{P}^{\leq N}$-Schur for any N. Now consider $\mathcal{F} = \mathcal{P}(^N X)$, or $\mathcal{P}^{\leq N}(X)$, or $\mathcal{P}(X)$. We say that X has the $\mathcal{F}$*-Dunford-Pettis property* ($\mathcal{F}$-DPP, for short) if it satisfies the equivalent conditions of the following theorem:

THEOREM 6.1. ([15, 82]) *Let $\mathcal{F} = \mathcal{P}(^N X)$, or $\mathcal{P}^{\leq N}(X)$, or $\mathcal{P}(X)$. The following conditions are equivalent:*

(a) For every Y, each weakly compact operator $X \to Y$ takes $\mathcal{F}$-null sequences in X into norm null sequences in Y.

(a') For every Y, each weakly compact polynomial $X \to Y$ takes $\mathcal{F}$-null sequences in X into norm convergent sequences in Y.

(b) For every weakly null sequence $(\phi_n) \subset X^*$ *and every* $\mathcal{F}$*-null sequence* $(x_n) \subset X$*, we have* $\phi_n(x_n) \to 0$.

(b') For each $m \in \mathbb{N}$*, for every weakly null sequence* $(P_n) \subset \mathcal{P}(^mX)$*, and every* $\mathcal{F}$*-null sequence* $(x_n) \subset X$*, we have* $P_n(x_n) \to 0$.

If $\mathcal{F} = \mathcal{P}^N$ *or* $\mathcal{P}^{\leq N}$*, these conditions are also equivalent to:*

(b") For every weakly null sequence $(P_n) \subset \mathcal{P}(^NX)$ *and every* $\mathcal{F}$*-null sequence* $(x_n) \subset X$*, we have* $P_n(x_n) \to 0$.

It is plain that both $\mathcal{P}^1$-DPP and $\mathcal{P}^{\leq 1}$-DPP coincide with the classical Dunford-Pettis property. In the case $N = 1$, the theorem is just Ryan's Theorem 5.3, and the general case is given in [15] (see also [37]). From the definition, we have that

$$\text{DPP} \Rightarrow \mathcal{P}^N\text{-DPP} \Rightarrow \mathcal{P}^{\leq N}\text{-DPP} \Rightarrow \mathcal{P}^{\leq N+1}\text{-DPP} \Rightarrow \mathcal{P}\text{-DPP}.$$

On the other hand, every $\mathcal{P}^{\leq N}$-Schur space (resp. Λ-space) has $\mathcal{P}^{\leq N}$-DPP (resp. $\mathcal{P}$-DPP). Further connections between these properties are given in the following two corollaries:

COROLLARY 6.1. ([20, 37]) *A Banach space is* $\mathcal{P}^{\leq N}$*-Schur if and only if it has* $\mathcal{P}^{\leq N}$*-DPP and is a* Λ*-space.*

COROLLARY 6.2. ([15])

(a) Suppose that X *is reflexive. Then* X *has* $\mathcal{P}^{\leq N}$*-DPP (resp.* $\mathcal{P}$*-DPP) if and only if* X *is* $\mathcal{P}^{\leq N}$*-Schur (resp. a* Λ*-space).*

(b) Suppose that $\mathcal{P}(X) = \mathcal{P}_{wsc}(X)$*. Then* X *has* $\mathcal{P}$*-DPP if and only if* X *has* $\mathcal{P}^{\leq N}$*-DPP, if and only if* X *has DPP.*

The spaces $\ell_p \oplus c_0$ and $\ell_p \oplus \ell_\infty$, where $1 \leq p < \infty$, have $\mathcal{P}^{\leq N}$-DPP for $N \geq p$, but are not $\mathcal{P}^{\leq M}$-Schur, for any M. More examples of spaces with $\mathcal{P}^{\leq N}$-Schur property, $\mathcal{P}^{\leq N}$-DPP, and $\mathcal{P}$-DPP are provided by the following results.

THEOREM 6.2. ([37])

(a) Suppose that X^* *contains no copy of* ℓ_1*, and* X^* *has property* (S_{p^*})*, where* $(1/p) + (1/p^*) = 1$*. Then* X *is* $\mathcal{P}^{\leq N}$*-Schur for* $N \geq p$.

(b) Suppose that X^* *has type* $p^* > 1$*, where* $(1/p) + (1/p^*) = 1$*. Then* X *is* $\mathcal{P}^{\leq N}$*-Schur for* $N > p$.

THEOREM 6.3. ([15]) *Suppose that* X *is* $\mathcal{P}^{\leq N}$*-Schur (resp. a* Λ*-space), and let* K *be any compact Hausdorff space. Then* $C(K, X)$ *has* $\mathcal{P}^{\leq N}$*-DPP (resp.* $\mathcal{P}$*-DPP).*

Talagrand has given examples of spaces $C(K, X)$ without the DPP [86]. Using Example 3.7 in [21], we can find a compact space K and a sequence (X_N) of spaces without the DPP such that each $C(K, X_N)$ fails the $\mathcal{P}^{\leq N}$-DPP.

Next we give some stability properties of spaces with $\mathcal{P}^{\leq N}$-DPP and $\mathcal{P}$-DPP.

THEOREM 6.4. ([15]) *Let* (X_n) *be a sequence of Banach spaces with* $\mathcal{P}$*-DPP (resp.* $\mathcal{P}^{\leq N}$*-DPP). Then the spaces* $\left(\bigoplus_n X_n\right)_{c_o}$ *and* $\left(\bigoplus_n X_n\right)_p$ *for* $1 \leq p < \infty$ *(resp.* $1 \leq p \leq N$*) have also* $\mathcal{P}$*-DPP (resp.* $\mathcal{P}^{\leq N}$*-DPP).*

When either $\mathcal{P}(X) = \mathcal{P}_{wsc}(X)$ or X is a Λ-space, then it is clear that the sum of two $\mathcal{P}$-null sequences in X is $\mathcal{P}$-null. However, for some spaces not satisfying these conditions, such as $c_0 \oplus \ell_2$, we still know that the same is true, thanks to the following result:

THEOREM 6.5. ([15]) *Suppose that* X *has the* $\mathcal{P}$*-DPP, and let* $(x_n), (y_n) \subset X$ *be* $\mathcal{P}$*-null sequences. Then* $(x_n + y_n)$ *is* $\mathcal{P}$*-null.*

As a different kind of polynomial property, we now turn our attention to the reflexivity of spaces of polynomials. We start with a fundamental result due to Ryan.

THEOREM 6.6. ([81]) *Suppose* X *has the approximation property, and let* N *be fixed. Then the following conditions are equivalent:*

(a) $\mathcal{P}(^N X)$ *is reflexive.*

(b) X *is reflexive and every* $P \in \mathcal{P}(^N X)$ *is weakly sequentially continuous.*

(c) Each bounded sequence in X *has a* $\mathcal{P}^{\leq N}$*-convergent subsequence.*

We do not know if the approximation property is necessary in the above Theorem.

If $\mathcal{P}(^N X)$ is reflexive for every N, then we say, according to Farmer [36], that X is *polynomially reflexive.* As noted in the comment after Theorem 3.3, T^* is polynomially reflexive. Other examples are provided by the following result due to Farmer:

THEOREM 6.7. ([36, 44]) *Suppose that* X *is reflexive and no normalized weakly null sequence in* X *has a lower* q*-estimate, for* $q > 1$*. Then* X *is polynomially reflexive.*

We now consider Q-reflexivity, a concept closely related to polynomial reflexivity, introduced by Aron and Dineen [7]. For the definition, we follow [55].

Given $z \in X^{**}$, we define an operator

$$\tilde{z} : \mathcal{L}_s(^N X) \longrightarrow \mathcal{L}_s(^{N-1}X)$$

by

$$\tilde{z}(L)(x_1, \dots, x_{N-1}) := \langle z, L(x_1, \dots, x_{N-1}, \cdot)\rangle .$$

If we now have N vectors $z_1, \dots, z_N \in X^{**}$, we consider the operators

$$\mathcal{L}_s(^N X) \xrightarrow{\tilde{z}_N} \mathcal{L}_s(^{N-1}X) \xrightarrow{\tilde{z}_{N-1}} \cdots \xrightarrow{\tilde{z}_1} \mathcal{L}_s(^0 X) = \mathbb{K}$$

and, given $L \in \mathcal{L}_s(^N X)$, we define

$$\bar{L}(z_1, \dots, z_N) := \tilde{z}_1 \circ \cdots \circ \tilde{z}_N(L) .$$

In this way, we obtain a continuous, N-linear functional $\bar{L}$ on X^{**} which extends L. Although the extended functional need not be symmetric, given $P \in \mathcal{P}(^N X)$, there is no ambiguity in defining its *Aron-Berner extension* $\bar{P}$ to X^{**} by $\bar{P}(z) := \bar{L}(z, \dots, z)$ where L is the symmetric, N-linear functional associated to P. The Aron-Berner extensions were introduced in [4], in a different way.

Thus we have a linear extension mapping from $\mathcal{P}(^N X)$ into $\mathcal{P}(^N X^{**})$. In fact, it has been proved in [23] that $\|\bar{P}\| = \|P\|$, so we can consider $\mathcal{P}(^N X)$ as a subspace of $\mathcal{P}(^N X^{**})$. For each $z \in X^{**}$ we have the evaluation functional $e_z \in \mathcal{P}(^N X)^*$ given by $e_z(P) := \bar{P}(z)$.

We now define the operator

$$\beta_N : \mathcal{P}(^N X)^{**} \longrightarrow \mathcal{P}(^N X^{**})$$

by

$$\beta_N(T)(z) := T(e_z) .$$

Then, β_N has norm one. According to [7], we say that X is *Q-reflexive* if β_N is a surjective isomorphism for each $N \in \mathbb{N}$. Note that if $\mathcal{P}(^N X)$ is reflexive, then β_N is the identity mapping and so, every polynomially reflexive space is Q-reflexive. It is shown in [7] that the non-reflexive Tsirelson*-James space T_J^* is Q-reflexive. The symmetric, projective tensor product of T_J^* with itself is shown in [40] to be an example of Q-reflexive, non-quasi-reflexive space.

A criterion for Q-reflexivity is given by the following result:

THEOREM 6.8. ([7, 55]) *Suppose X^{**} has the Radon-Nikodym property and the approximation property. Then:*
(a) β_N is surjective;
(b) β_N is injective if and only if $\mathcal{P}(^N X) = \mathcal{P}_{wsc}(^N X)$.

Combining this with the results of section 4 yields:

COROLLARY 6.3. *Suppose X^{**} has the Radon-Nikodym property and the approximation property. Then:*
(a) If no normalized weakly null sequence in X admits a lower q-estimate, then β_N is a surjective isomorphism for all $N < q$.
(b) If X^* has property (S_{p^*}) with $(1/p) + (1/p^*) = 1$, then β_N is not injective for $N \geq p$.

We know of no example of a space such that β_N fails to be surjective. It is shown in [55] that if X^{**} has the bounded approximation property (see [63, Definition 1.e.11]), then β_N is surjective for all $N \in \mathbb{N}$.

In connection with Q-reflexivity, conditions for spaces of polynomials to have the Radon-Nikodym property are given in [7] and [90]. For further information about the bidual of spaces of polynomials, we refer to [55] and [90].

Properties such as containment or finite representability of ℓ_∞ in spaces of polynomials are investigated in [29] and [30]. We finish this section with two polynomial properties introduced by González and Gutiérrez in [42] and [43]. Recall that a series $\sum_{i=1}^{\infty} x_i$ is *unconditionally convergent* (u.c., for short) if every subseries is norm convergent. A polynomial $P \in \mathcal{P}(^k X; Y)$ $(k \in N)$ is *unconditionally converging* if P maps w.u.C. series into u.c. series. We say that X is a *Grothendieck space* if every weak-star null sequence in X^* is weakly null. Similarly, X is said to be $\mathcal{P}^N$*-Grothendieck* if every pointwise null sequence in $\mathcal{P}(^N X)$ is weakly null. On the other hand, X has Pełczyński's *property (V)* if, for every Y, each unconditionally converging operator $X \to Y$ is weakly compact. Finally, X is said to be $\mathcal{P}^N$-(V) if, for every Y, each N-homogeneous, unconditionally converging polynomial $X \to Y$ is weakly compact. The main result concerning these properties is the following:

THEOREM 6.9. ([42, 43]) *For a Banach space X and fixed $N \geq 2$, the following statements are equivalent:*
(a) X is $\mathcal{P}^N$-Grothendieck.
(b) X is $\mathcal{P}^N$-(V).
(c) $\mathcal{P}(^N X)$ is reflexive.

For a different approach to some of the polynomial properties studied in this section, we refer to [18].

7. Properties (P) and (RP)

In relation with the PDPP and Λ-spaces, Aron, Choi and Llavona [5] introduced properties (P) and (RP) for polynomials. The motivation was the following:

If $g : B_{c_0} \to B_{c_0}$ is defined by $g((x_n)) = (x_n^n)$, then g is not uniformly continuous, since

$$\left\| g(e_n) - g\left[\left(1 - \frac{1}{n}\right) e_n\right]\right\| \longrightarrow 1 - e^{-1} > 0\,.$$

However, using Pełczyński's result cited in the introduction to section 4, $P \circ g$ is uniformly continuous for every continuous polynomial P on c_0. Suppose $g : B_X \to B_Y$ is a function that satisfies property (*) below:

(*) For any polynomial $P \in \mathcal{P}(Y)$, the composition $P \circ g : B_X \to \mathbb{K}$ is uniformly continuous.

Is g necessarily uniformly continuous?

This problem will be solved if we are able to determine when the following condition is satisfied:

(**) Given two bounded sequences $(u_n), (v_n) \subset Y$, if $P(u_n) - P(v_n) \to 0$ for every $P \in \mathcal{P}(Y)$, then it follows that $u_n - v_n \to 0$.

We introduce the following condition which, as we shall see, is strictly weaker than (**). We say that Y has *property* (P) if whenever two bounded sequences $(u_j), (v_j) \subset Y$ satisfy $|P(u_j) - P(v_j)| \to 0$ for every $P \in \mathcal{P}(Y)$, then it follows that $Q(u_j - v_j) \to 0$ for every $Q \in \mathcal{P}(Y)$.

In addition, the study of property (P) naturally leads to the following reciprocal property, which is also studied in this section. We say that Y has *property* (RP) if whenever two bounded sequences $(u_j), (v_j) \subset Y$ satisfy $P(u_j - v_j) \to 0$ for every $P \in \mathcal{P}(Y)$, then it follows that $Q(u_j) - Q(v_j) \to 0$ for every $Q \in \mathcal{P}(Y)$.

We begin with the following easy facts about property (P):

Proposition 7.1. ([5])

(a) Every Banach space with the DPP has property (P).

(b) Let $g : B_X \to B_Y$ be a function that satisfies property (*). If Y is a Λ-space with property (P), then g is uniformly continuous.

If every scalar valued polynomial on X is weakly sequentially continuous at the origin, then X has property (P). Thus, for instance, the space T^* has property (P). Although the spaces $L^1[0,1]$, c_0 and ℓ_∞ satisfy property (P), they do not satisfy condition (**).

An interesting result on property (P) is the following:

THEOREM 7.1. ([15]) *Suppose X^* has property (S_p) for some $1 < p < \infty$ and X^* contains no copy of ℓ_1. Then X has property (P).*

Since every superreflexive space is in W_p for some $1 < p < \infty$, it satisfies the conditions of the Theorem, and so every superreflexive space has property (P). In particular, every $L^p(\mu)$ space has property (P), for $1 < p < \infty$.

Choi and Kim [22, Theorem 2.1] have shown that every space with non-trivial type has property (P). No example is known of a Banach space failing property (P). Moreover, although it is obvious that every Banach space with property (**) described above is a Λ-space, we do not know if the reverse implication holds. However, every Λ-space with property (P) satisfies (**).

In relation to property (RP), it is elementary to show that every Λ-space, and c_0 have property (RP). In fact, we have:

THEOREM 7.2. ([5])

(a) Every Λ-space has property (RP).

(b) c_0 has property (RP).

(c) If X has DPP and contains no copy of ℓ_1, then X has property (RP).

(d) The spaces $\ell_\infty (\simeq L_\infty[0,1])$, $L_1[0,1]$ and $C[0,1]$ do not have property (RP).

We sketch the proof that the space $L_\infty[0,1]$ fails to have property (RP). Let $I_j := (1/2^j, 1/2^{j-1})$ ($j = 1, 2, \dots$), and let $(r_j(t))_{j=1}^\infty$ be the Rademacher functions on $[0,1]$. Define

$$A : L_\infty[0,1] \times L_\infty[0,1] \times L_\infty[0,1] \longrightarrow \mathbb{K}$$

by

$$A(f,g,h) := \sum_{j=1}^\infty \left(2^j \int_{I_j} f(t)\, dt \right) \langle g, r_j \rangle \langle h, r_j \rangle ,$$

where $\langle g, r_j \rangle = \int_0^1 g(t) r_j(t)\, dt$. It is easy to see that A is a continuous, trilinear form. For each j, let $s_j = \chi_{I_j}$, $x_j = r_j$ and $y_j = r_j + s_j$. Since (s_j) is weakly null in $L_\infty[0,1]$, the DPP of $L_\infty[0,1]$ implies that $P(x_j - y_j) \to 0$ for every continuous, scalar valued polynomial P. Let Q be the 3-homogeneous polynomial given by $Q(f) = A(f, f, f)$, for $f \in L_\infty[0,1]$. Since

$$\int_{I_j} r_n = \begin{cases} 0 & \text{if } 1 \leq j \leq n-1 \\ -2^{-j} & \text{if } j = n \\ 2^{-j} & \text{if } j > n \end{cases}$$

we see that $Q(r_n) = -1$ for all n. Similarly, $Q(y_n) = 0$, and so $|Q(x_n) - Q(y_n)| = 1$, which completes the proof.

It is worth noting that $Q(x_j) - Q(y_j) \to 0$ for every 2-homogeneous polynomial Q on $L_\infty[0,1]$. In fact, suppose that (x_j) and (y_j) are bounded sequences in ℓ_∞ such that $(x_j - y_j)$ is weakly null. Let $Q \in \mathcal{P}(^2\ell_\infty)$ and let A be the unique continuous symmetric, bilinear form associated to Q, so that $Q(x) = A(x,x)$. We can regard A as the bounded operator $A : x \in \ell_\infty \mapsto A(x, \cdot) \in \ell_\infty^*$. Since ℓ_∞^* contains no copy of c_0, it follows that $A : \ell_\infty \to \ell_\infty^*$ is weakly compact [80]. By the DPP of ℓ_∞, we get that $A(x_j - y_j) \to 0$ in ℓ_∞^*. Then it follows that $Q(x_j) - Q(y_j) \to 0$.

8. Polynomial continuity

A mapping $f : X \to Y$ is said to be *polynomially continuous* (*P-continuous*, for short) if its restriction to any bounded set is uniformly continuous for the weak polynomial topology, i.e., if for every $\epsilon > 0$ and bounded $B \subset X$, there are a finite set $\{P_1, \ldots, P_n\} \subset \mathcal{P}(X)$ and $\delta > 0$ so that $\|f(x) - f(y)\| < \epsilon$ whenever $x, y \in B$ satisfy $|P_j(x-y)| < \delta$ $(1 \leq j \leq n)$. Clearly, if every scalar valued polynomial on X is P-continuous, then X enjoys property (RP). In the proof of Theorem 7.2(d) given in [5], polynomials on $L_\infty[0,1]$, $L^1[0,1]$ and $C[0,1]$ are constructed which are not P-continuous. For certain spaces, such as c_0, every P-continuous mapping is weakly uniformly continuous on bounded sets, but this is not true in general. For example, $\|x\|^2$ is clearly P-continuous on the real space ℓ_2, although it is not weakly uniformly continuous on the ball. On the other hand, every P-continuous mapping is uniformly continuous on bounded sets. It is easy to see that the norm is not P-continuous on c_0, and that, for some Banach spaces, such as ℓ_2, uniform continuity on bounded sets and P-continuity coincide.

A characterization of the spaces for which uniform continuity and P-continuity coincide is given in the following theorem:

THEOREM 8.1. ([49]) *For a real Banach space X, the following assertions are equivalent:*

(a) The wp and the norm topologies coincide on X.
(b) X has a separating polynomial.
(c) Every uniformly continuous, real valued function on X is P-continuous.
(d) The norm is P-continuous on X.

Recall that a polynomial $P \in \mathcal{P}(X)$ is separating if $P(0) = 0$ and $P(x) \geq 1$ for every $x \in X$ with $\|x\| = 1$. Satisfy the above theorem the spaces $L^p(\mu)$ with p an even integer, and the products of these spaces. Before studying the P-continuity, we shall give a few results on the existence of separating polynomials.

This separation property was first used by Kurzweil [59] in order to obtain the following approximation result:

THEOREM 8.2. ([59]) *Suppose that X is a separable, real Banach space which admits a separating polynomial. Then every continuous real valued function on X can be approximated, uniformly on X, by real analytic functions on X.*

We do not know whether the same result is true for nonseparable spaces.

Geometrical conditions for the existence of a separating polynomial are given in [33] and [25]. If a real Banach space X admits a separating polynomial, then X is superreflexive (see [25, Theorem 2] and [34, Theorem 3.3]). Deville [24] proved that if X admits a separating polynomial then X contains an isomorphic copy of ℓ_{2n} for some $n \in \mathbb{N}$. This was made more precise in [46], in connection with the upper and lower indices of the space:

THEOREM 8.3. ([24, 46]) *Suppose that X is a real Banach space which admits a separating polynomial. Then $u(X) = 2m$ and $l(X) = 2n$, for some $n, m \in \mathbb{N}$, and X contains an isomorphic copy of both ℓ_{2n} and ℓ_{2m}.*

In fact, when the space X has a *subsymmetric basis* (i.e., a basis which is equivalent to any of its subsequences), something stronger can be said:

THEOREM 8.4. ([46]) *Suppose that X is a real Banach space with a subsymmetric basis, which admits a separating polynomial. Then X is isomorphic to ℓ_{2n} for some $n \in \mathbb{N}$.*

We now study some properties of P-continuous operators. Since an operator is compact if and only if it is weakly (uniformly) continuous on bounded sets [11, Proposition 2.5], every compact operator is P-continuous.

It can be proved that every P-continuous operator is weakly compact. The proof needs a description of the polynomials on ℓ_1, using the notation of [83]. We write $N_k^{(N)}$ for the set of multi-indices of degree k, i.e., the set of sequences $m = (m_j)_{j=1}^{\infty}$, with $m_j \in \mathbb{N}$, and $\sum_{j=1}^{\infty} m_j = k$. We let $m! = \prod_{j=1}^{\infty} m_j!$, where the usual convention $0! = 1$ is observed. If $a = (a_j)$ is a sequence of scalars, then $a^m := \prod_{j=1}^{\infty} a_j^{m_j}$, where 0^0 is defined to be 1.

LEMMA 8.1. ([49]) *Every $P \in \mathcal{P}(^k\ell_1)$ may be written in the form $P(t) = \sum_{m \in N_k^{(N)}} a_n t^n$, for $t \in \ell_1$, with scalar coefficients a_m satisfying the estimate*

$$|a_m| \frac{m^m}{k^k} \leq C_k \|P\| ,$$

for some constant $C_k > 0$ depending on. k. If ℓ_1 is complex, we may take $C_k = 1$, and in the real case, $C_k = (2k)^k/k!$.

Using this Lemma and Ramsey's theorem [56, Lemma 29.1], it is possible to prove:

LEMMA 8.2. ([49]) *Given $P \in \mathcal{P}(^k\ell_1)$ with k even, and $\epsilon > 0$, there are $n \in \mathbb{N}$ and $t \in \ell_1$, $\|t\| = 1$, so that $|P(t)| < \epsilon$ and*

$$t = \frac{1}{2N} \left(e_{p_1} + \cdots + e_{p_N} - e_{p_{N+1}} - \cdots - e_{p_{2N}}\right)$$

where $p_1 < \cdots < p_{2N}$.

We now sketch the proof of the next theorem.

THEOREM 8.5. ([49]) *Every P-continuous operator is weakly compact.*

Proof. Let $T : X \to Y$ be a P-continuous operator, and assume it is not weakly compact. We can find operators $U : \ell_1 \to X$, $S : \ell_1 \to \ell_\infty$ and $V : Y \to \ell_\infty$ with $S((t_n)) = \left(\sum_{i=1}^{n} t_i\right)_n$, so that $VTU = S$ [62, Theorem 8.1]. Then S is P-continuous.

There is a wp-null net in the unit sphere of ℓ_1, with elements of the form

$$t = \frac{1}{2N} \left(e_{p_1} + \cdots + e_{p_N} - e_{p_{N+1}} - \cdots - e_{p_{2N}}\right) . \tag{1}$$

Indeed, given a finite set of homogeneous polynomials $\{P_1, \dots , P_n\} \subset \mathcal{P}(\ell_1)$, if ℓ_1 is constructed over the real field, we set $P := P_1^{\alpha_1} + \cdots + P_n^{\alpha_n}$

so that P is a homogeneous polynomial of even degree. By Lemma 8.2, given $\epsilon > 0$, there is $t \in S_{\ell_1}$ of the form (1) so that $|P(t)| < \epsilon$.

Since $\|S(t)\| = 1/2$, S is not wp-continuous on the unit ball, a contradiction.

If ℓ_1 is complex, we then need an easy adaptation of Lemma 8.2 for a finite set of polynomials that may be assumed of even degree. ∎

However, not every P-continuous polynomial is weakly compact; an example is the polynomial $P \in \mathcal{P}({}^k\ell_2, \ell_1)$, given by $P((t_n)) = \left(t_n^k\right)_n$.

As we mentioned in the introduction to this section, if every scalar valued polynomial on X is P-continuous, then X has property (RP). We can ask whether the converse is true. We show that the answer is no, by giving examples of polynomials on both ℓ_1 and ℓ_3 which are not P-continuous.

It is easy to show that a polynomial P is P-continuous if and only if so is the associated operator T_P. Let $P \in \mathcal{P}({}^2\ell_1)$ be given by

$$P(t) = \sum \{t_j t_k : j \text{ even}, 1 \le k < j\}$$

for $t = (t_i) \in \ell_1$. Since the associated operator T_P is not weakly compact [6, p. 83], this is an example of a polynomial on ℓ_1 which is not P-continuous.

In the case of ℓ_3, it is easy to show that the 3-homogeneous polynomial given by $P(x) := \sum_{i=1}^\infty x_i^3$, for $x = (x_i) \in \ell_3$, is not P-continuous.

Since $\mathcal{P}({}^2\ell_3) = \mathcal{P}_{wb}({}^2\ell_3)$ (see the Introduction to section 4), from Theorem 3.2, we get that every 2-homogeneous polynomial on ℓ_3 is P-continuous.

By Theorem 8.1, we know that the wp-topology and the norm topology coincide on a real Hilbert space. From this, we have

THEOREM 8.6. ([49]) *Every scalar valued 2-homogeneous polynomial on a real $C(K)$ space is P-continuous.*

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