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Some Remarks on the Christoffel Symbols of Ehresmann ϵ-Connections

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We study the relations between torsion, curvature, local flatness and simplicity within the framework of ϵ-connections by means of explicit formulas in terms of the Christoffel symbols of the given ϵ-connection.

1. Introduction

Let M be a differentiable manifold of dimension n and $S\widehat{F}^k(M) \to M$ be the semi-holonomic frame bundle of M of order k. An ϵ-connection of order k is a $GL_1(n, \Re)$-invariant section of $S\widehat{F}^k(M) \to \widehat{F}^1(M)$ (our convention with the order is different than the one in [10], [11] for notational convenience here). They are first defined in [2] and are studied in [5], [10], [11]. It is shown in [2] that there is a bijection between ϵ-connections of order k and linear connection of order $k-1$. An ϵ-connection (linear connection) is called simple in [10], [11] if it is obtained from a first order ϵ-connection (linear connection) by successive prolongations. It is known ([10], [11]) that a linear connection is locally flat if and only if it is simple and without torsion and curvature. Even though the concepts of curvature, torsion and local flatness are well known within the framework of linear connections, simplicity is relatively less known. This latter concept is used in [3] in relation to mechanics to study second grade materials. The same problem is studied in [1] by means of the inhomogeneity tensor.

This note is resulted from an attempt to understand the interactions between torsion, curvature, local flatness and simplicity by means of explicit formulas in terms of Christoffel symbols in the spirit of classical differential geometry. The reason for our choice of ϵ-connections is that, as we show, their Christoffel symbols are conceptually very simple objects compared to those of linear connections. We define in this note also a dual ϵ-connection and give a necessary condition for a dual ϵ-connection of arbitrary order to be obtained from successive prolongations of a section of first order coframes, in a sense to be made precise below. We also give a characterization of (dual) ϵ-connections which are prolongations of a second order (first order in [10], [11]) flat ϵ-connection. Our formulas are expressed explicitly in terms of the Christoffel symbols of the given ϵ-connection. As an interesting fact, the classical curvature tensor emerges in our formulas in a very natural way (see (9) and (14)). The usage of two principle bundles here corresponds to fixing the source and the targets of jets in groupoids. Further, there seems to be an intimate relation between the formulas here and the first nonlinear Spencer sequence defined in [9]. We hope to clarify this relation in some future work. For details on this latter sequence and its applications in various branches of mathematical physics, we refer the reader to, for instance, [8].

2. Christoffel Symbols of ϵ-Connections and Dual ϵ-Connections

Let M be a differentiable manifold of dimension n and $\widehat{F}^k(M) \to M$ be the frame bundle of M of order k. The elements of $\widehat{F}^k(M)$ are k-jets of local diffeomorphisms with source at the origin of $\Re^n$ and target in M. $\widehat{F}^k(M) \to M$ is a right principal bundle with group $GL_k(n,\Re)$. For the details, we refer the reader to [10], [11] and the references therein. In this note, we will need also the coframe bundle $\widetilde{F}^k(M) \to M$. The elements of $\widetilde{F}^k(M)$ are k-jets of local diffeomorphisms with source in M and target at the origin of $\Re^n$. $\widetilde{F}^k(M) \to M$ is a left principal bundle with group $GL_k(n,\Re)$. There exists a bijective map $\rho : \widehat{F}^k(M) \to \widetilde{F}^k(M)$ with the property $\rho(ua) = a^{-1}\rho(u)$, $u \in \widehat{F}^k(M)$, $a \in GL_k(n,\Re)$. Let $\widehat{\Gamma}$ be a $GL_1(n,\Re)$-invariant section of $\widehat{F}^k(M) \to \widehat{F}^1(M)$. Such sections, called ϵ-connections, are defined in [2] in the framework of semi-holonomic frame bundles and are studied in [2], [5], [10], [11]. They are known to be in one-to-one correspondence with torsionfree linear connections on $\widehat{F}^{k-1}(M) \to M$, $k \geq 2$ (see [4]). We will define here also a dual ϵ-connection $\widetilde{\Gamma}$ as an $GL_1(n,\Re)$-invariant section of $\widetilde{F}^k(M) \to \widetilde{F}^1(M)$. Now let $(x^i, \widehat{x}^i_j, \ldots, \widehat{x}^i_{j_1 \ldots j_k})$, $(x^i, \widetilde{x}^i_j, \ldots, \widetilde{x}^i_{j_1 \ldots j_k})$ be local coordinates on $\widehat{F}^k(M)$, $\widetilde{F}^k(M)$ and $\widehat{\Gamma}$, $\widetilde{\Gamma}$ be two such invariant sections. A straightforward computation using $GL_1(n,\Re)$-invariance shows that there existst functions $\widehat{\Gamma}^i_{j_1 j_2}(x), \ldots, \widehat{\Gamma}^i_{j_1 \ldots j_k}(x)$ and $\widetilde{\Gamma}^i_{j_1 j_2}(x), \ldots, \widetilde{\Gamma}^i_{j_1 \ldots j_k}(x)$, called the Christoffel symbols of $\widehat{\Gamma}$ and $\widetilde{\Gamma}$, which determine $\widehat{\Gamma}$ and $\widetilde{\Gamma}$ locally by the formulas

$$\widehat{\Gamma}(x,\widehat{x})^i_{j_1\dots j_m} = \widehat{\Gamma}^i_{s_1\dots s_m}(x)\widehat{x}^{s_1}_{j_1}\dots\widehat{x}^{s_m}_{j_m} \tag{1}$$

$$\widetilde{\Gamma}(x,\widehat{x})^i_{j_1\dots j_m} = \widetilde{x}^i_s\widetilde{\Gamma}^s_{j_1\dots j_m}(x)\,, \qquad \text{for} \quad 2\le m\le k\,. \tag{2}$$

See also [10] for (1). It follows from (1) and (2) that both sets of Christoffel symbols have consistent transformation rules under a coordinate change $\phi : (x) \to (y)$. Let $p : GL_k(n,\Re) \to GL_1(n,\Re)$ be the projection homomorphism where we regard $GL_1(n,\Re)$ as a subgroup of $GL_k(n,\Re)$ by the injection $GL_1(n,\Re) \to (GL_1(n,\Re),0,\dots,0)$. Using the more suggestive notation $\left(\frac{\partial y}{\partial x}\right)$ for the section $j_k(\phi)$, we have

PROPOSITION 1. *The Christoffel symbols* $\widehat{\Gamma}^i_{j_1j_2}(x),\dots,\widehat{\Gamma}^i_{j_1\dots j_k}(x)$ *and* $\widetilde{\Gamma}^i_{j_1j_2}(x),\dots,\widetilde{\Gamma}^i_{j_1\dots j_k}(x)$ *are subject to the transformation rules*

$$\left(\frac{\partial y}{\partial x}\right)\left(\delta^i_j,\widehat{\Gamma}^i_{j_1j_2}(x),\dots,\widehat{\Gamma}^i_{j_1\dots j_k}(x)\right)p\left(\frac{\partial y}{\partial x}\right)^{-1} = \left(\delta^i_j,\widehat{\Gamma}^i_{j_1j_2}(y),\dots,\widehat{\Gamma}^i_{j_1\dots j_k}(y)\right) \tag{3}$$

$$p\left(\frac{\partial y}{\partial x}\right)^{-1}\left(\delta^i_j,\widetilde{\Gamma}^i_{j_1j_2}(x),\dots,\widetilde{\Gamma}^i_{j_1\dots j_k}(x)\right)\left(\frac{\partial x}{\partial y}\right) = \left(\delta^i_j,\widetilde{\Gamma}^i_{j_1j_2}(y),\dots,\widetilde{\Gamma}^i_{j_1\dots j_k}(y)\right) \tag{4}$$

where we are using the group operation of $GL_k(n,\Re)$ *in the formulas* (3) *and* (4).

The proof of Proposition 1 follows in a straightforward way from (1) and (2). Note that classical Christoffel symbols emerge from (4) for $k=2$. Comparing (3) and (4), we obtain

COROLLARY 1. *Let* $\widehat{\Gamma}$ *be an* ϵ*-connection of order* k *with Christoffel symbols* $\widehat{\Gamma}^i_{j_1j_2}(x),\dots,\widehat{\Gamma}^i_{j_1\dots j_k}(x)$. *Then, the functions* $\widetilde{\Gamma}^i_{j_1j_2}(x),\dots,\widetilde{\Gamma}^i_{j_1\dots j_k}(x)$ *defined by*

$$\left(\delta^i_j,\widetilde{\Gamma}^i_{j_1j_2}(x),\dots,\widetilde{\Gamma}^i_{j_1\dots j_k}(x)\right) = \left(\delta^i_j,\widehat{\Gamma}^i_{j_1j_2}(x),\dots,\widehat{\Gamma}^i_{j_1\dots j_k}(x)\right)^{-1} \tag{5}$$

determine a dual ϵ*-connection* $\widetilde{\Gamma}$ *with Christoffel symbols* $\widetilde{\Gamma}^i_{j_1j_2}(x),\dots,\widetilde{\Gamma}^i_{j_1\dots j_k}(x)$.

Let $\widehat{\epsilon}^k(M)$ and $\widetilde{\epsilon}^k(M)$ be the spaces of ϵ-connections and dual ϵ-connections of order k. The following result generalizes geodesic coordinates to higher order Christoffel symbols.

Proposition 2. *Let $\widehat{\Gamma} \in \widehat{\epsilon}^k(M)$ and $p \in M$. Then there exists a coordinate neighbourhood U containing p such that all the Christoffel symbols of $\widehat{\Gamma}$ on U vanish at p. The analogous statement holds for $\widetilde{\Gamma} \in \widetilde{\epsilon}^k(M)$.*

The proof of Proposition 2 is immediate from (3) and (4). Note that if (x) is geodesic for $\widehat{\Gamma}$ and $\widetilde{\Gamma}$ at p and (y) is arbitrary, then the Christoffel symbols of $\widehat{\Gamma}$ and $\widetilde{\Gamma}$ at p with respect to (y) are given, respectively, by

$$\frac{\partial^s y^i}{\partial x^{m_1} \cdots \partial x^{m_s}} \frac{\partial x^{m_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{m_s}}{\partial y^{j_s}} \quad \text{and} \quad \frac{\partial y^i}{\partial x^m} \frac{\partial^s x^m}{\partial y^{j_1} \cdots \partial y^{j_s}}.$$

Thus we see that the Christoffel symbols of (dual) ϵ-connections are conceptually very simple objects compared to those of linear connections on frame and coframe bundles which are related to differential forms and Lie algebras. In fact, they are nothing but sections of frame or coframe bundles over M (holonomic, semi-holonomic, ..., see below) twisted by their first order jets from right or left.

Also, it follows from (5) that $\widehat{\Gamma}^i_{jk} = -\widetilde{\Gamma}^i_{jk}$, but clearly $\widehat{\Gamma}^i_{j_1 \ldots j_s} \neq -\widetilde{\Gamma}^i_{j_1 \ldots j_s}$ for $s \geq 3$ in general. A careful examination of the derivations of the classical formulas now shows, in our opinion, that $\xi_{i;k} = \xi_{i,k} + \widehat{\Gamma}^m_{ki}\xi_m$ and $\xi_{i;k} = \xi^i_{,k} + \widetilde{\Gamma}^i_{km}\xi^m$, that is, classical covariant differentiation is somehow related to inversion in $GL_2(n, \Re)$.

Now, $\widehat{\epsilon}^k(M)$ and $\widetilde{\epsilon}^k(M)$ can be identified with the associated bundles of $\widehat{F}^k(M) \to M$ and $\widetilde{F}^k(M) \to M$ where the left and the right actions of $GL_k(n, \Re)$ on the kernel $(1, B^n_2, \ldots, B^n_k)$ of the projection homomorphism $GL_k(n, \Re) \to GL_1(n, \Re)$ are given by the formulas (3) and (4) respectively. With these identifications, Christoffel symbols become local coordinates on the natural bundles $\widehat{\epsilon}^k(M) \to M$ and $\widetilde{\epsilon}^k(M) \to M$ and $\widehat{\Gamma}, \widetilde{\Gamma}$ become sections of these bundles. Note that the kernel $(1, B^n_2, \ldots, B^n_k)$ can be identified with the left/right coset spaces $L(GL_k(n, \Re)/GL_1(n, \Re))$, $R(GL_k(n, \Re)/GL_1(n, \Re))$ and the above actions become standart actions of $GL_k(n, \Re)$ on these coset spaces. Consequently, there is a one-to-one correspondence between the sections of $\widehat{\epsilon}^k(M) \to M$, $\widetilde{\epsilon}^k(M) \to M$ and $GL_1(n, \Re)-$reductions on $\widehat{F}^k(M) \to M$, $\widetilde{F}^k(M) \to M$. Also the bijection $\rho : \widehat{F}^k(M) \to \widetilde{F}^k(M)$ induces a bijection $\widehat{\epsilon}^k(M) \to \widetilde{\epsilon}^k(M)$ which is given locally by (5).

In the rest of this note we will concentrate on coframes but our arguments remain valid for frames. The main reason for this is that the classical curvature tensor, like classical Christoffel symbols, emerges in a natural way from second order coframes and not from frames, contrary to one might expect.

In order to drop the symmetry condition on the Christoffel symbols, let $SGL_k(n,\Re)$ be the semi-holonomic jet group of order k and $s\widetilde{\epsilon}^k(M)$ be the associated bundle of $\widetilde{F}^k(M) \to M$ with respect to the action of $GL_k(n,\Re)$ on the kernel of $SGL_k(n,\Re) \to GL_1(n,\Re)$ determined by (4). Clearly all the above arguments remain valid for $s\widetilde{\epsilon}^k(M)$, except Proposition 2, which makes essential use of symmetry. If $\widetilde{\Gamma} \in \widetilde{\epsilon}^k(M)$, then $\widetilde{\Gamma}$ will be called symmetric. It can be shown that $\widetilde{\Gamma}$ is symmetric if and only if the corresponding linear connection is quasi-holonomic and torsionfree (see [10] for a proof in the case of $\widehat{\Gamma}$). Let $S\widetilde{F}^k(M) \to M$ be the semi-holonomic coframe bundle of M of order k. If (U,x^i) is a coordinate neighbourhood and $(f^i_j(x),\ldots,f^i_{j_1\ldots j_k}(x))$is a section of $S\widetilde{F}^k(M) \to M$ on (U,x^i), using the same notation for bundles and their sheaves of sections, we define a map $S\widetilde{F}^k(M) \to s\widetilde{\epsilon}^k(M)$ locally by $(f^i_j,\ldots,f^i_{j_1\ldots j_k}) \to (1, g^i_m f^m_{j_1j_2},\ldots,g^i_m f^m_{j_1\ldots j_k})$ where $g^i_m f^m_j = \delta^i_j$ and we omit x. It follows from (4) that this map does not depend on the choice of coordinates and is locally surjective by the definition of $s\widetilde{\epsilon}^k(M)$. We also have a map $\widetilde{F}^1(M) \to J^{k-1}\,\widetilde{F}^1(M)$ which is locally given by $(f^i_j) \to (f^i_j, \partial_{j_1} f^i_{j_2},\ldots,$ $\partial_{j_1}\cdots\partial_{j_{k-1}} f^i_{j_k})$. Indeed, differentiating $f^i_m(x)\frac{\partial x^m}{\partial y^j} = f^i_j(y)$ $k-1$ times, we obtain $(j_{k-1}f)(y) * j_k(\phi) = (j_{k-1}f)(x)$ where $*$ denotes the group operation in $SGL_k(n,\Re)$ and ϕ denotes the coordinate change $(x) \to (y)$. The last equality shows that $J^{k-1}\,\widetilde{F}^1(M)$ can be identified with a subset of $S\widetilde{F}^k(M)$. Composing this map with the above map, we obtain a differential operator $\widetilde{v} : \widetilde{F}^1(M) \to s\widetilde{\epsilon}^k(M)$ of order $k-1$ which is locally given by $(f^i_j) \to$ $(1, g^i_m\partial_{j_1} f^m_{j_2},\ldots,g^i_m\partial_{j_1}\cdots\partial_{j_{k-1}} f^m_{j_k})$. In particular, if $\widetilde{\Gamma}^i_{j_1\ldots j_s}$ is symmetric in the indices $j_1,\ldots,j_{s-1}$ for $s \geq 3$, then this property does not depend on the choice of the coordinates and defines a subbundle $s_2\widetilde{\epsilon}^k(M)$ of $s\widetilde{\epsilon}^k(M)$ and we have $\mathrm{Im}\widetilde{v} \subseteq s_2\widetilde{\epsilon}^k(M)$.

Now let $\mathrm{Diff}(M,\Re^n)$ be the set of local diffeomorphisms from M into $\Re^n$. We also have the map $\widetilde{j}_1 : \mathrm{Diff}(M,\Re^n) \to \widetilde{F}^1(M)$ which is locally given by $\widetilde{j}_1(f^i) = \partial_j f^i$.

To recapitulate, we have

PROPOSITION 3. *We have a sequence of maps*

$$\mathrm{Diff}(M,\Re^n) \xrightarrow{\widetilde{j}_1} \widetilde{F}^1(M) \xrightarrow{\widetilde{v}} s_2\widetilde{\epsilon}^k(M) \tag{6}$$

where $\widetilde{j}_1(f^i) = \partial_j f^i$, $\widetilde{v}(f^i_j) = (1, g^i_m\partial_{j_1} f^m_{j_2},\ldots,g^i_m\partial_{j_1}\cdots\partial_{j_{k-1}} f^m_{j_k})$ *and* $g^i_m f^m_j = \delta^i_j$.

DEFINITION. $\widetilde{\Gamma} \in s\widetilde{\epsilon}^k(M)$ will be called locally flat if the Christoffel symbols of $\widetilde{\Gamma}$ vanish identically in some coordinates.

The following proposition, whose proof is identical to the proof of Proposition 2, gives a simple characterization of locally flat ϵ-connections.

PROPOSITION 4. *$\widetilde{\Gamma} \in s\widetilde{\epsilon}^k(M)$ is locally flat if and only if $\widetilde{\Gamma} \in \mathrm{Im}(\widetilde{v} \circ \widetilde{j}_1)$.*

Now let $\widetilde{\Gamma} \in s_2\widetilde{\epsilon}^k(M)$. There exists some $f \in \widetilde{F}^1(U)$ such that $\widetilde{v}(f) = \widetilde{\Gamma}_U$, if and only if we can solve the nonlinear system of PDE

$$g^i_m \partial_{j_1} \ldots \partial_{j_{s-1}} f^m_{j_s} = \widetilde{\Gamma}^i_{j_1 \ldots j_s}\,, \qquad 2 \leq s \leq k\,, \tag{7}$$

for the functions f^i_j. The equations (7) have the necessary integrability conditions

$$\widetilde{d}_{[j_1} \widetilde{\Gamma}^i_{j_2] \ldots j_{s+1}} = 0 \tag{8}$$

where the differential expressions $\widetilde{d}_{j_1} \widetilde{\Gamma}^i_{j_2 \ldots j_{s+1}}$ are defined by

$$\widetilde{d}_{j_1} \widetilde{\Gamma}^i_{j_2 \ldots j_{s+1}} = \frac{\partial \widetilde{\Gamma}^i_{j_2 \ldots j_{s+1}}}{\partial x^{j_1}} + \widetilde{\Gamma}^i_{j_1 m} \widetilde{\Gamma}^m_{j_2 \ldots j_{s+1}} \tag{9}$$

Note that $\widetilde{d}_{[j} \widetilde{\Gamma}^i_{k]l}$ gives the classical curvature tensor R^i_{ljk}. For $k = 2$, the operator $\widetilde{v}$ is first order and (8) is also sufficient to solve $\widetilde{v}(f^i_j) = g^i_m \partial_{j_1} f^m_{j_2} = \widetilde{\Gamma}^i_{j_1 j_2}$ for f^i_j. If $\widetilde{\Gamma}$ is further symmetric, then we can solve $\widetilde{v} \circ j_1(f^i) = g^i_m \partial_{j_1} \partial_{j_2} f^m = \widetilde{\Gamma}^i_{j_1 j_2}$ for f^i and we recover the classical fact that $\widetilde{\Gamma}$ is locally flat if and only if it is symmetric and without curvature. Note that (8) can be derived also from the first order system

$$g^i_m \partial_{j_1} \cdots \partial_{j_{s-1}} f^m_{j_s} = \widetilde{\Gamma}^i_{j_1 \ldots j_s}\,, \quad 2 \leq s \leq k\,, \tag{10}$$

for the functions $f^i_j, \ldots, f^i_{j_1 \ldots j_{k-1}}$. As an unpleasent fact now, the local formula $(f^i_j, \ldots, f^i_{j_1 \ldots j_{k-1}}) \to (1, g^i_m \partial_{j_1} f^m_{j_2}, \ldots, g^i_m \partial_{j_1} f^i_{j_2 \ldots j_k})$ does not give a well defined operator $\widetilde{F}^{k-1}(M) \to s\widetilde{\epsilon}^k(M)$ for $k \geq 3$. Consequently, one faces the following problem which we hope to clarify in some future work.

PROBLEM. Do the integrability conditions (8) have any intrinsic meaning for $k \geq 3$?

In any case we will use (9) to give a characterization of ϵ-connections which are prolongations of a second order ϵ-connection without curvature and give an affirmative answer to the above problem in a special case. For $m \leq k$ let $\pi^k_m : s_2\widetilde{\epsilon}^k(M) \to s_2\widetilde{\epsilon}^m(M)$ be the projection map.

PROPOSITION 5. *Let $\widetilde{\Gamma} \in s_2\widetilde{\epsilon}^k(M)$ have Christoffel symbols $\widetilde{\Gamma}^i_{jk}$ and suppose that $\widetilde{\Gamma}$ has no curvature, that is,* (8) *holds on M with $k = 2$. Then the functions defined by*

$$\widetilde{\Sigma}^i_{j_1 \dots j_s} = d_{j_1 \dots j_{s-2}} \widetilde{\Gamma}^i_{j_{s-1} j_s}\,, \quad 2 \le s \le k\,, \tag{11}$$

are the Christoffel symbols of some element of $s_2\widetilde{\epsilon}^k(M)$, which will be denoted by $P^k(\widetilde{\Gamma})$. Further, $P^k(\widetilde{\Gamma}) \in \operatorname{Im}(\widetilde{v})$ and is locally flat if $\widetilde{\Gamma}$ is symmetric. Conversely, if $\widetilde{\Sigma} \in \operatorname{Im}(\widetilde{v})$, then $P^k(\pi^k_2(\widetilde{\Sigma})) = \widetilde{\Sigma}$.

The proof of Proposition 5 is easy using (7). It can be shown that $P^k(\widetilde{\Gamma})$ in the above proposition coincides with the simple ϵ-connection determined by $\widetilde{\Gamma}$ in the sense of [10].

We will now slightly modify Proposition 5. It is easy to see that $s\widetilde{\epsilon}^k(M) \to M$ is an affine bundle. Let $E^k(M) \to M$ be the model vector bundle with local coordinates $\lambda^i_{j_1 j_2}, \dots, \lambda^i_{j_1 \dots j_k}$ and $E^{k-s+1}(M) \to M$ be a subbundle of $E^k(M) \to M$ defined by $\lambda^i_{j_1 j_2} = \dots = \lambda^i_{j_1 \dots j_s} = 0$ for $2 \le s \le k$. Then $\{0\} = E^1(M) \subseteq E^2(M) \subseteq \dots \subseteq E^{k-1}(M) \subseteq E^k(M)$ and $E^2(M)$ is the $(1,k)$ tensor bundle $T^1_k(M)$ of M. Finally, let $fs\widetilde{\epsilon}^k(M) = \{\widetilde{\Gamma} \in s\widetilde{\epsilon}^k(M) : \pi^k_2(\widetilde{\Gamma})$ has no curvature$\}$.

PROPOSITION 6. *There exists a differential operator $\widetilde{D} : fs\widetilde{\epsilon}^k(M) \to E^{k-1}(M)$ of order $k-2$ which is defined by $D(\widetilde{\Gamma}) = \widetilde{\Gamma} - P^k(\pi^k_2(\widetilde{\Gamma}))$. In particular, if $\pi^k_s(\widetilde{\Gamma}) = P^s(\pi^k_2(\widetilde{\Gamma}))$, then $D(\widetilde{\Gamma}) \in E^{k-s+1}(M)$.*

We refer to [7] for an exact differential sequence which extends Propositions 3 and 6 above.

Now, the expressions given by (9) can be formally derived as follows ([6]). Let $\widetilde{A}_s$ be the formal equality given by

$$\widetilde{A}_s : \quad \widetilde{\Gamma}^i_{j_1 \dots j_s}(x) = \frac{\partial x^i}{\partial z^m} \frac{\partial^s z^m}{\partial x^{j_1} \cdots \partial x^{j_s}}\,. \tag{12}$$

It is now clear that both sides of (13) have the same transformation rules which are given by (4), an observation which has been our starting point. Differentiating $\widetilde{A}_s$ with respect to x^r and formally substituting $\widetilde{A}_2$ into $\widetilde{A}_{s+1}$ and changing indices, we obtain (9). As an amusing fact, this derivation requires no knowledge whatsoever except differentiation. Clearly, this formal computation amounts to working with flat coordinates z^i. There is also a trick which is dual to the one above: Let $\widehat{B}_s$ be the formal equality given by

$$\widehat{B}_s : \quad \widehat{\Gamma}^i_{j_1 \dots j_s}(x) = \frac{\partial^s x^i}{\partial z^{m_1} \cdots \partial z^{m_s}} \frac{\partial z^{m_1}}{\partial x^{j_1}} \cdots \frac{\partial z^{m_s}}{\partial x^{j_s}}\,. \tag{13}$$

Differentiating (14) with respect to x^r and substituting $\widehat{B}_2$ and $\widehat{B}_{s+1}$, we obtain the differential expressions

$$\widehat{d}_{j_1}\widehat{\Gamma}^i_{j_2\cdots j_{s+1}} = \frac{\partial\widehat{\Gamma}^i_{j_2\cdots j_{s+1}}}{\partial x^{j_1}} + \widehat{\Gamma}^m_{j_1j_2}\widehat{\Gamma}^i_{mj_3\cdots j_{s+1}} + \cdots + \widehat{\Gamma}^m_{j_1j_{s+1}}\widehat{\Gamma}^i_{j_2\cdots j_s m}\,. \qquad (14)$$

Note that $\widehat{d}_{[j}\widehat{\Gamma}^i_{k]l}$ already incorporates the classical torsion $\widehat{\Gamma}^i_{[jk]}$. If $\widehat{\Gamma}^i_{[jk]} = 0$ and we substitute $\widehat{\Gamma}^i_{jk} = -\widetilde{\Gamma}^i_{jk}$ in $\widehat{d}_{[j}\widehat{\Gamma}^i_{k]l}$, we obtain $-\widetilde{d}_{[j}\widetilde{\Gamma}^i_{k]l}$. Note also the interesting relation between (9) and (14) in view of covariant differentiation and $\widehat{\Gamma}^i_{jk} = -\widetilde{\Gamma}^i_{jk}$. Clearly, there seems to be no conceptual difference between (9) and (14).

Now the case of frames can be studied along similar lines (with some bad surprises along the way) but we will omit the details.

We will end with the following

PROBLEM. What is the relation of (9) and (14) to the operator $\overline{D}'$ in the first nonlinear Spencer sequence

$$0 \to \mathrm{Aut}(M) \xrightarrow{j_{k+1}} \Pi_{k+1}(M,M) \xrightarrow{\overline{D}} T^* \otimes J_k(T) \xrightarrow{\overline{D}'} \Lambda^2(T^*) \otimes J_{k-1}(T)\,?$$

Note that this sequence incorporates frames and coframes at the same time. We refer to, for instance, [8] for the details on this sequence.

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