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Non-Holonomic Mechanical Systems in Jet Bundles *

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In this paper we present a geometrical formulation for Lagrangian systems subjected to non-holonomic constraints in terms of jet bundles. Cosymplectic geometry and almost product structures are used to obtain the constrained dynamics without using Lagrange multipliers method.

1. Introduction

The natural arena for studying time-dependent mechanical systems is a fibered manifold $\pi : E \longrightarrow \mathbb{R}$ (see [14, 6, 8, 22] and references therein). In fact, E is the configuration space, and the evolution space is $J^1\pi$, the manifold of 1-jet of local sections of π. The Lagrangian is a function $L : J^1\pi \longrightarrow \mathbb{R}$. As we have shown in [12, 4, 8], the use of almost cosymplectic geometry instead of symplectic geometry permits us to derive in a very geometrical setting the motion equations.

Non-holonomic constrained systems have a long subject of research since the first times of Mechanics (see, for instance, [15] and [14] for a large bibliography on this subject) and they have deserved great attention in recent years [23, 24, 13, 17, 18, 14, 1, 2, 7, 5, 19, 3, 20, 21, 9, 10, 11].

In this paper, we suppose that the Lagrangian L is subjected to a set of constraints $\{\phi_i\}$ which are affine in the velocities, that is, $\phi_i = (\mu_i)_A(t,q)\dot{q}^A + h_i(t,q)$. An almost product structure $(\mathcal{P}, \mathcal{Q})$ is defined on the manifold $J^1\pi$ such that the projection $\mathcal{P}(\xi_L)$ of the Euler-Lagrange vector field ξ_L corresponding to the free problem gives the dynamics of the constrained problem.

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2. Jet Manifolds

Let E be an $(n+1)$-dimensional fibered manifold over $\mathbb{R}$, i. e., there exists a surjective submersion

$$\pi : E \longrightarrow \mathbb{R} .$$

The 1-jet manifold $J^1\pi$ is defined as the space of 1-jets of local sections of π, i. e.,

$$J^1\pi = \left\{ \begin{array}{ccc} j^1_t\phi & / & \phi : U \subset \mathbb{R} \longrightarrow E \,, \pi \circ \phi = id_U \\ & & U \text{ open neighborhood of } t \end{array} \right\} .$$

If (t, q^A) are fibered coordinates on E, then $J^1\pi$ has local coordinates $(t, q^A, \dot{q}^A)$. In fact, if $\phi(s) = (s, \phi^A(s)), s \in U$, $j^1_t\phi$ is determined by

$$(t, \phi^A(t), \frac{d\phi^A}{dt}(t)) .$$

So $J^1\pi$ is a $(2n+1)$-dimensional manifold. We also deduce that $J^1\pi$ is a fibered manifold over E and $\mathbb{R}$, with canonical projections $\pi_{1,0} : J^1\pi \longrightarrow E$ and $\pi_1 : J^1\pi \longrightarrow \mathbb{R}$. In local coordinates we have

$$\begin{aligned} \pi_{1,0}(t, q^A, \dot{q}^A) &= (t, q^A) \,, \\ \pi_1(t, q^A, \dot{q}^A) &= t \,, \\ \pi(t, q^A) &= t \,. \end{aligned}$$

Notice that we can choose fibered coordinates such that t is the standard coordinate on $\mathbb{R}$.

There exists a canonical embedding of $J^1\pi$ into TE. In fact, any 1-jet $j^1_t\phi$ defines a tangent vector $\dot{\phi}(t) \in T_{\phi(t)}E$. In local coordinates we have

$$(t, q^A, \dot{q}^A) \longmapsto (t, q^A, 1, \dot{q}^A) .$$

Remark. If E is the trivial fibration $\pi : \mathbb{R} \times Q \longrightarrow \mathbb{R}$, then $J^1\pi$ may be identified with $\mathbb{R} \times TQ$.

There exists a canonical tensor field $\tilde{J}$ of type $(1,1)$ on $J^1\pi$ defined as follows. Let be $\tilde{X} \in T_{j^1_t\phi}(J^1\pi)$ and take its projections to E and $\mathbb{R}$:

$$T\pi_{1,0}(\tilde{X}) \in T_{\phi(t)}E \,, \quad T\pi_1(\tilde{X}) \in T_t\mathbb{R} .$$

Hence

$$T\pi_{1,0}(\tilde{X}) - T\phi(t)T\pi_1(\tilde{X}) \in (V\pi)_{\phi(t)} \,,$$

where $V\pi$ is the vertical subbundle of π-vertical tangent vectors on E. Take its vertical lift to TE and its restriction to $J^1\pi$, since it is tangent to $J^1\pi$. Hence

$$\tilde{J}\tilde{X} = \left(T\pi_{1,0}(\tilde{X}) - T\phi(t)T\pi_1(\tilde{X})\right)^v_{/J^1\pi} .$$

In local coordinates we obtain

$$\left.\begin{array}{l} \tilde{J}(\frac{\partial}{\partial t}) = -\dot{q}^A \frac{\partial}{\partial \dot{q}^A} \\ \tilde{J}(\frac{\partial}{\partial q^A}) = \frac{\partial}{\partial \dot{q}^A} \\ \tilde{J}(\frac{\partial}{\partial \dot{q}^A}) = 0 \end{array}\right\}$$

or, equivalently,

$$\tilde{J} = (dq^A - \dot{q}^A dt) \otimes \frac{\partial}{\partial \dot{q}^A} .$$

In a similar way, we define the manifold of 2-jets of local sections $J^2\pi$ with local coordinates $(t, q^A, \dot{q}^A, \ddot{q}^A)$. Notice that $J^2\pi$ is a fibered manifold over $J^1\pi$, E and $\mathbb{R}$ with canonical projections

$$\begin{aligned} \pi_{2,1} &: J^2\pi \longrightarrow J^1\pi , \\ \pi_{2,0} &: J^2\pi \longrightarrow E , \\ \pi_2 &: J^2\pi \longrightarrow \mathbb{R}. \end{aligned}$$

There exists a natural inclusion of $J^2\pi$ into the 1-jet manifold $J^1\pi_1$. In fact, define

$$\begin{aligned} \gamma \ : \ & J^2\pi \hookrightarrow J^1\pi_1 \\ & j^2_t\phi \longmapsto j^1_t\psi , \end{aligned}$$

where $\psi(s) = j^1_s\phi$. In local coordinates, γ reads as

$$\gamma(t, q^A, \dot{q}^A, \ddot{q}^A) = (t, q^A, \dot{q}^A, \dot{q}^A, \ddot{q}^A) .$$

Moreover, there exists a natural embedding of $J^1\pi_1$ into $TJ^1\pi$. So, we have the following inclusions

$$J^2\pi \stackrel{\gamma}{\hookrightarrow} J^1\pi_1 \stackrel{u}{\hookrightarrow} TJ^1\pi .$$

We shall consider a special class of vector fields on $J^1\pi$. We say that a vector field ξ on $J^1\pi$ is a non-autonomous second order differential equation

(NSODE for simplicity) if $\xi : J^1\pi \longrightarrow TJ^1\pi$ takes values into $(u \circ \gamma)(J^2\pi)$. Therefore, ξ is an NSODE iff it has the following local expression:

$$\xi(t, q^A, \dot{q}^A) = \frac{\partial}{\partial t} + \dot{q}^A \frac{\partial}{\partial q^A} + \xi^A \frac{\partial}{\partial \dot{q}^A} \, .$$

If we put $\eta = (\pi_1)^*(dt)$, we obtain the following geometrical characterization of an NSODE: ξ is an NSODE iff $\tilde{J}(\xi) = 0$ and $\eta(\xi) = 1$.

A local section ϕ of $\pi : E \longrightarrow \mathbb{R}$ may be considered as a curve in E. We denote by $j^1\phi$ the 1-jet prolongation of ϕ to $J^1\pi$ which can also be viewed as a curve in the manifold $J^1\pi$. We say that ϕ is a solution of an NSODE ξ if its 1-jet prolongation $j^1\phi$ is an integral curve of ξ. Thus, $\phi(t) = (t, \phi^A(t))$ is a solution of ξ iff it satisfies the following system of non-autonomous differential equations of second order:

$$\left.\begin{array}{l} \ddot{q}^A = \xi^A(t, q, \frac{dq}{dt}) \\ \dot{q}^A = \frac{d\phi^A}{dt} \end{array}\right\} .$$

(Notice that any integral curve of ξ is a 1-jet prolongation).

3. Non-Holonomic Lagrangian Dynamics

A C^∞ function $L : J^1\pi \longrightarrow \mathbb{R}$ is said to be a non-autonomous or time-dependent Lagrangian. Associated with L we define the Poincaré-Cartan 1-form

$$\Theta_L = L\eta + \tilde{J}^*(dL)$$

and the Poincaré-Cartan 2-form

$$\Omega_L = -d\Theta_L \, .$$

In local coordinates we have

$$\begin{array}{rcl} \Theta_L & = & (L - \dot{q}^A \hat{p}_A)dt + \hat{p}_A dq^A \, , \\ \Omega_L & = & d(\dot{q}^A \hat{p}_A - L) \wedge dt + dq^A \wedge d\hat{p}_A \, , \end{array}$$

where $\hat{p}_A = \dfrac{\partial L}{\partial \dot{q}^A}$ are the generalized momenta.

We say that L is regular if the Hessian matrix

$$\left(\frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} \right)$$

is non-singular. So, L is regular iff (Ω_L, η) is an almost cosymplectic structure on $J^1\pi$. In that case, there exists a unique vector field ξ_L on $J^1\pi$ such that

$$i_{\xi_L}\Omega_L = 0 \ , \quad i_{\xi_L}\eta = 1 \ . \tag{1}$$

ξ_L is the Reeb vector field of the almost cosymplectic structure (Ω_L, η). Moreover:

1. ξ_L is a NSODE.
2. The solutions of ξ_L are just the solutions of the Euler-Lagrange equations for L:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^A}\right) - \frac{\partial L}{\partial q^A} = 0 \ , \quad 1 \leq A \leq n \ . \tag{2}$$

Next, we shall suppose that the Lagrangian L is subjected to m independent non-holonomic constraints $\{\phi_i \ ; \ 1 \leq i \leq m\}$, $m \leq n$, given by

$$\phi_i = (\mu_i)_A(t,q)\dot{q}^A + h_i(t,q) \ .$$

The constraint functions ϕ_i may be intrinsically defined as follows. Take m linearly independent 1-forms on E $\{\mu_i \ ; \ 1 \leq i \leq m\}$ and the complete lift μ_i^c to TE. If $\mu_i = (\mu_i)_A dq^A + h_i dt$, we get

$$\begin{aligned} \mu_i^c \ = \ & \tau \frac{\partial(\mu_i)_A}{dt} dq^A + \tau \frac{\partial h_i}{\partial t} dt + \dot{q}^B \frac{\partial(\mu_i)_A}{\partial q^B} dq^A \\ & + \dot{q}^B \frac{\partial h_i}{\partial q^B} dt + (\mu_i)_A d\dot{q}^A + h_i d\tau \ , \end{aligned}$$

where $(t, q^A, \tau, \dot{q}^A)$ are the induced coordinates on TE. Define the 1-forms $\bar{\mu}_i$ on $J^1\pi$ by

$$\bar{\mu}_i = \tilde{J}^*(\mu^c_{i/J^1\pi}) \ .$$

Hence we obtain

$$\bar{\mu}_i = (\mu_i)_A dq^A - \dot{q}^A(\mu_i)_A dt \ .$$

Moreover, define the function $\hat{\mu}_i \ : \ TE \longrightarrow \mathbb{R}$ by

$$\hat{\mu}_i(X) = \langle \mu_i(x), X \rangle \ , \quad \forall X \in T_xE \ .$$

So, we have

$$\hat{\mu}_i(t, q^A, \tau, \dot{q}^A) = (\mu_i)_A \dot{q}^A + h\tau \ .$$

Thus, its restriction to $J^1\pi$ is just ϕ_i, say $\phi_i(t, q^A, \dot{q}^A) = \hat{\mu}_i(t, q^A, 1, \dot{q}^A)$.

In order to obtain independent constraint functions we have to assume that dt and $\{\mu_i\}$ are linearly independent. The motion is allowable on the submanifold P_1 defined by the vanishing of the constraint functions ϕ_i. So, the motion equations (1) have to be modified:

$$\left.\begin{array}{rcl} i_X\Omega_L & = & \lambda^i\bar{\mu}_i \\ i_X\eta & = & 1 \\ X(\phi_i) & = & 0 \end{array}\right\} \tag{3}$$

where λ^i are Lagrange multipliers to be computed. In fact, if X is a solution of (3) we deduce that X is an NSODE and its solution satisfy the Euler-Lagrange equations with constraints:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^A}\right) - \frac{\partial L}{\partial q^A} = -\lambda^i(\mu_i)_A\ ,\ 1 \leq A \leq n\ . \tag{4}$$

Define the vector field Z_i by

$$i_{Z_i}\Omega_L = \bar{\mu}_i\ , \quad i_{Z_i}\eta = 0\ . \tag{5}$$

Then $\tilde{J}Z_i = 0$ and Z_i is a $\pi_{1,0}$-vertical vector field. Define the matrix $\mathcal{C}$ of order m whose entries are

$$\mathcal{C}_{ij} = Z_i(\phi_j)\ .$$

If $\mathcal{C}$ is regular we define a tensor field $\mathcal{Q}$ of type $(1,1)$ by

$$\mathcal{Q} = \mathcal{C}^{ij} Z_j \otimes d\phi_i\ ,$$

where $(\mathcal{C}^{ij})$ is the inverse matrix of $\mathcal{C}$. A direct computation shows that $\mathcal{Q}^2 = \mathcal{Q}$. Thus $(\mathcal{P} = id_{J^1\pi} - \mathcal{Q}, \mathcal{Q})$ is an almost product structure on $J^1\pi$. In principle, $(\mathcal{P}, \mathcal{Q})$ is only defined on the points of P_1, but from regularity of $\mathcal{C}$, we conclude that $(\mathcal{P}, \mathcal{Q})$ is also defined on some open neighborhood of P_1. It is obvious that

$$i_{\mathcal{P}\xi_L}\Omega_L = 0\ , \quad i_{\mathcal{P}\xi_L}\eta = 0\ .$$

Moreover, we have

$$\begin{array}{rcl} \mathcal{P}(\xi_L)(\phi_k) & = & \xi_L(\phi_k) - \mathcal{C}^{ij}\xi_L(\phi_i)Z_j(\phi_k) \\ & = & \xi_L(\phi_k) - \mathcal{C}^{ij}\xi_L(\phi_i)\mathcal{C}_{jk} \\ & = & \xi_L(\phi_k) - \delta^i_k\xi_L(\phi_i) \\ & = & 0\ , \end{array}$$

which implies that $\mathcal{P}(\xi_L)$ is tangent to P_1.

We conclude that $\mathcal{P}(\xi_L)$ is the solution of the dynamics, the Lagrange multipliers being

$$\Lambda^j = -\mathcal{C}^{ij}\xi_L(\phi_i) , \quad 1 \leq j \leq m ,$$

Notice that since L is regular, the unique solution of (3) is just $\mathcal{P}(\xi_L)$.

4. The Singular Case

Suppose now that the matrix $\mathcal{C}$ is singular (with constant rank) on P_1. From (3) and (5) we get $X = \xi_L + \lambda^i\bar{\mu}_i$. So $X(\phi_i) = 0$ may be equivalently written as

$$\xi_L(\phi_j) + \lambda^i Z_i(\phi_j) = 0 . \tag{6}$$

Consider the set P_2 of all the points of P_1 where this system of equations have solutions (λ^i) and suppose that P_2 is a submanifold of P_1. Consistency of the dynamics impose that the solutions verifying (6) of (3) must be tangent to P_2. Nevertheless, these solutions on P_2 may not be tangent to P_2. Then, we have to restrict P_2 to a submanifold where the solutions of (3) are tangent to P_2. Proceeding further we obtain a sequence of constraint submanifolds

$$\cdots \to P_k \to \cdots \to P_2 \to P_1 .$$

We have three possibilities:

1. There exists an integer $k > 1$ such that $P_k = \emptyset$. This means that the equations (3) are not consistent.

2. There exists an integer $k > 1$ such that $P_k \neq \emptyset$ but $\dim P_k = 0$ In this case, there is no dynamics. P_k consists in isolated points and the solution of the dynamics is $X = 0$.

3. There exists an integer $k > 1$ such that $P_{k+1} = P_k$ and $\dim P_k > 0$. In such a case, there exists at least a solution ξ on P_k verifying the NSODE condition (that is, $(\tilde{J}\xi = 0)_{/P_k}$ and $(i_\xi\eta = 1)_{/P_k}$) such that:

$$\left(i_\xi\Omega_L = \lambda^i\bar{\mu}_i\right)_{/P_k} .$$

In the third case, the manifold P_k is called the final constraint submanifold and it will be denoted by P_f.

Suppose that P_f is given by the vanishing of constraints $\chi_{i'}$, $(1 \leq i' \leq m')$, where $m' \geq m$. Construct the matrix $\mathcal{C}' = (Z_{i'}(\chi_{j'}))$, $(1 \leq i' \leq m')$, $(1 \leq j' \leq m')$ and suppose that it has contant rank, i.e., $rank\ \mathcal{C}' = k' < m$ $(k \leq k')$. For simplicity, we suppose that the matrix $\mathcal{C}'_1 = (\mathcal{C}'_{i'j'}) = (Z_{i'}(\chi_{j'}))$, $(1 \leq i', j' \leq k')$, is non-singular. As in the regular case, we construct an almost product structure $(\mathcal{P}, \mathcal{Q})$ given by

$$\mathcal{Q} = (\mathcal{C}')^{i'j'} Z_{j'} \otimes d\chi_{i'} \ , \quad (1 \leq i', j' \leq k' < m) \ ,$$

and $\mathcal{P} = id - \mathcal{Q}$, where $(\mathcal{C}')^{i'j'}$ is the $i'j'$-entry of the inverse matrix of $\mathcal{C}'_1$. For each solution $X = \xi_L + \lambda^l Z_l + \lambda^\alpha Z_\alpha$, $(1 \leq l \leq k',\ k'+1 \leq \alpha \leq m)$ of (3) we obtain that $\mathcal{P}(X)$ is a solution of the motion equations on P_f. In fact, we have

$$\mathcal{P}(X) = \xi_L - (\mathcal{C}')^{i'j'} \left(\xi_L(\chi_{i'}) + \lambda^\beta Z_\beta(\chi_{i'})\right) Z_{j'} + \lambda^\alpha Z_\alpha \ ,$$

with $k'+1 \leq \beta \leq m$. We then conclude that

$$\begin{cases} \left(i_{\mathcal{P}(X)}\Omega_L = \lambda^i \bar{\mu}_i\right)_{/P_f} \\ \left(i_{\mathcal{P}(X)}\eta = 1\right)_{/P_f} \\ \left((\mathcal{P}(X))(\chi_{i'}) = 0\right)_{/P_f} \end{cases} .$$

5. The Hamiltonian Formalism

Let $L : J^1\pi \longrightarrow \mathbb{R}$ be a regular Lagrangian function. In order to obtain the Hamiltonian counterpart of the dynamics, we first define the Legendre transformation associated with L.

Define

$$Leg : J^1\pi \longrightarrow T^*E$$

as follows: $Leg(j^1_t\phi)$ is the 1-form at the point $\phi(t)$ such that

$$\langle Leg(j^1_t\phi), X\rangle = \langle(\Theta_L)(j^1_t\phi), \tilde{X}\rangle \ ,$$

where $X \in T_{\phi(t)}E$ and $\tilde{X}$ is an arbitrary tangent vector at $j^1_t\phi$ which projects onto X, i.e., $T\pi_{1,0}(\tilde{X}) = X$. In local coordinates we obtain:

$$Leg(t, q^A, q^A_1) = (t, q^A, L - \dot{q}^A\hat{p}_A, \hat{p}_A) \ .$$

Now, put

$$(T^*_u E)_x = \{\alpha \in T^*_x E \ / \ i_u\alpha = 0, \forall u \in (V\pi)_x\} \ .$$

Then, $T_v^*E = \cup_{x \in E}(T_v^*E)_x$ is a vector subbundle of $\pi_E : T^*E \longrightarrow E$ of rank 1. Denote the quotient vector bundle by $J^1\pi^*$, say

$$J^1\pi^* = T^*E/T_v^*E \; .$$

$J^1\pi^*$ is a vector bundle over E of rank n and we have and exact sequence of vector bundles over E

$$0 \longrightarrow T_v^*E \xrightarrow{\iota} T^*E \xrightarrow{\nu} J^1\pi^* \longrightarrow 0 \tag{7}$$

If we consider local coordinates (t, q^A, p_t, p_A) on T^*E, then we have local coordinates (t, q^A, p_t) on T_v^*E and (t, q^A, p_A) on $J^1\pi^*$. Notice that $J^1\pi^*$ is a vector bundle over E and a fibered manifold over $\mathbb{R}$, with canonical projections:

$$\pi_{1,0}^* : J^1\pi^* \longrightarrow E$$

and

$$\pi_1^* : J^1\pi^* \longrightarrow \mathbb{R} \; ,$$

respectively.

Denote by $leg : J^1\pi \longrightarrow J^1\pi^*$ the composition of leg with ν, i.e.,

$$leg = \nu \circ Leg \; .$$

Since L is regular, Leg is an inmersion and leg a local diffeomorphism. Assume, for the sake of simplicity, that L is hyperregular, that is, $leg : J^1\pi \longrightarrow J^1\pi^*$ is a global diffeomorphism. In such a case, we define a global section $h : J^1\pi^* \longrightarrow T^*E$ of the canonical projection $\nu : T^*E \longrightarrow J^1\pi^*$ by putting

$$h = Leg \circ leg^{-1} \; .$$

(If L is regular we only have local sections of ν). h will be called a Hamiltonian.

Define the 2-form $\Omega_h = h^*\omega_E$ on $J^1\pi^*$, where ω_E is the canonical symplectic form on T^*E and $\eta_1 = (\pi_1^*)^*(dt)$. A simple computation shows that:

1. (Ω_h, η_1) is an almost cosymplectic structure on $J^1\pi^*$.
2. Denote by E_h the Reeb vector field for (Ω_h, η_1), i. e., $i_{E_h}\Omega_h = 0$, $i_{E_h}\eta_1 = 1$, and suppose that

$$h(t, q^A, p_A) = (t, q^A, H(t, q^A, p_A), p_A) \; .$$

Hence the projections onto E of the integral curves of E_h satisfy the Hamilton equations:

$$\frac{dq^A}{dt} = -\frac{\partial H}{\partial p_A} \; , \quad \frac{dp_A}{dt} = \frac{\partial H}{\partial q^A} \; .$$

3. (Ω_L, η) and (Ω_h, η_1) are *leg*-related, i.e., $leg^*\Omega_h = \Omega_L$, $leg^*\eta_1 = \eta$.

4. As a consequence, ξ_L and E_h are also *leg*-related.

Now, suppose that L is subjected to non-holonomic constraints given by a family of linearly independent 1-forms μ_i on E. We define the 1-forms $\tilde{\mu}_i$ and the functions $\tilde{\phi}_i$ on $J^1\pi^*$ by

$$\begin{aligned} \tilde{\mu_i} &= (leg^{-1})^*(\bar{\mu_i}) , \\ \tilde{\phi}_i &= \phi_i \circ leg^{-1} . \end{aligned}$$

Thus, the Hamilton equations for the constrained system are:

$$\left.\begin{aligned} i_{\tilde{X}}\Omega_h &= \tilde{\lambda}^i\tilde{\mu}_i \\ i_{\tilde{X}}\eta_1 &= 1 \\ \tilde{X}(\tilde{\phi}_i) &= 0 \end{aligned}\right\} \tag{8}$$

where $\tilde{\lambda}^i$ are Lagrange multipliers to be determined.

The almost product structure $(\mathcal{P}, \mathcal{Q})$ on $J^1\pi$ is transported onto an almost product structure $(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}})$ on $J^1\pi^*$ such that

$$\tilde{Q} = \tilde{\mathcal{C}}^{ij}\tilde{Z}_j \otimes d\tilde{\phi}_i ,$$

being $\tilde{\mathcal{C}}^{ij}$ the entries of the inverse matrix of $\tilde{\mathcal{C}} = (\tilde{\mathcal{C}}^{ij} = \tilde{Z}_i(\tilde{\phi}_j))$ and $\tilde{Z}_i$ the $\pi^*_{1,0}$-vertical vector fields given by

$$\left.\begin{aligned} i_{\tilde{Z}_i}\Omega_h &= \tilde{\mu}_i \\ i_{\tilde{Z}_i}\eta_1 &= 0 \end{aligned}\right\} .$$

Of course, $\tilde{Z}_i$ and Z_i are *leg*-related. As a conclusion, we obtain that $\tilde{\mathcal{P}}(E_h)$ is the solution of the motion equations (8) with Lagrange multipliers

$$\tilde{\Lambda}^j = -\tilde{\mathcal{C}}^{ij}E_h(\tilde{\phi}_i) ,$$

because $\mathcal{P}(\xi_L)$ and $\tilde{\mathcal{P}}(E_h)$ are *leg*-related.

In a similar way, if we apply the algorithm developed in Section 4 to the motion equations (8), we obtain a sequence of constraints submanifolds $\tilde{M}_i$ of $J^1\pi^*$. These submanifolds $\tilde{M}_i$ are precisely $\tilde{M}_i = leg(P_i)$. Thus, both algorithms are connected by means of the Legendre transformation *leg*. If one of them stabilizes, the other one too, and conversely.

6. An Example

Suppose that a point A moves on the axis Ox, the distance OA being a prescribed function $f(t)$ of t. The particle of mass m, whose position at time t is (x, y), moves in the xy-plane, and is constrained so that at each instant its velocity is directed towards A. This curve is called *curve of pursuit* (see [16]).

Consider the trivial bundle $\pi : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$, $\pi(t, x, y) = t$ and the jet bundle $J^1\pi$ with coordinates $(t, x, y, \dot{x}, \dot{y})$. We can describe this system by the Lagrangian $L : J^1\pi \longrightarrow \mathbb{R}$:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) .$$

and the constraint

$$\phi = y\dot{x} + (f(t) - x)\dot{y} = 0 .$$

A direct computation shows that

$$\begin{aligned} \Theta_L &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)dt + m\dot{x}dx + m\dot{y}dy , \\ \Omega_L &= m\dot{x}d\dot{x} \wedge dt + m\dot{y}d\dot{y} \wedge dt + mdx \wedge d\dot{x} + mdy \wedge d\dot{y} , \\ \xi_L &= \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} . \end{aligned}$$

Consider the 1-form $\mu = ydx + (f(t) - x)dy$. We have that

$$\bar{\mu} = ydx + (f(t) - x)dy - \dot{x}ydt - \dot{y}(f(t) - x)dt .$$

The vector field Z such that $i_Z\Omega_L = \bar{\mu}$ and $i_Z dt = 0$ is

$$Z = -\frac{y}{m}\frac{\partial}{\partial \dot{x}} - \frac{(f(t) - x)}{m}\frac{\partial}{\partial \dot{y}} .$$

Now,

$$\mathcal{C} = -\frac{1}{m}(y^2 + (f(t) - x)^2) ,$$

and then the proyector $\mathcal{Q}$ is given by

$$\begin{aligned} \mathcal{Q} &= \mathcal{C}^{-1} Z \otimes d\phi \\ &= \frac{1}{y^2 + (f(t) - x)^2}\left(y\frac{\partial}{\partial \dot{x}} + (f(t) - x)\frac{\partial}{\partial \dot{y}}\right) \\ &\quad \otimes \left(\dot{x}dy + yd\dot{x} + \frac{\partial f}{\partial t}\dot{y}dt - \dot{y}dx + (f(t) - x)d\dot{y}\right) . \end{aligned}$$

Finally, the solution of the constrained dynamics is the vector field

$$\mathcal{P}(\xi_L) = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} - \frac{y\dot{y}}{y^2 + (f(t)-x)^2}(\frac{\partial f}{\partial t})\frac{\partial}{\partial \dot{x}} - \frac{(f(t)-x)\dot{y}}{y^2 + (f(t)-x)^2}(\frac{\partial f}{\partial t})\frac{\partial}{\partial \dot{y}} .$$

So, the solutions of the dynamics are the ones of the following system of non-autonomous second order differential equations

$$\begin{cases} \ddot{x} &= -\dfrac{y\dot{y}}{y^2 + (f(t)-x)^2}\dfrac{\partial f}{\partial t} \\ \ddot{y} &= -\dfrac{(f(t)-x)\dot{y}}{y^2 + (f(t)-x)^2}\dfrac{\partial f}{\partial t} . \end{cases}$$

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