EXTRACTA MATHEMATICAE Vol. 11, Núm. 1, 116 – 126 (1996)

Geometric Characterization of the Homogeneity of Continua with Microstructure †

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AMS *Subject Class.* (1991): 73B05, 73B10, 73B25, 53C10

1. Introduction

A continuum with microstructure may geometrically be modelled as an associated bundle with a principal bundle. The homogeneity is characterized by using the theory of connections in principal bundles.

A continuum with microstructure is a simple body $\mathcal{B}$ each point of which has attached a manifold of parameters [2]. In geometrical terms, we have a body manifold $\mathcal{B}$ and a fiber bundle $\tilde{\pi} : \tilde{\mathcal{E}} \longrightarrow \mathcal{B}$ over $\mathcal{B}$. Some kind of homogeneity is needed for each fiber and the geometrical measure of this homogeneity is supplied by the action of a Lie group on the manifold of parameters. In geometrical words, the fiber bundle is associated with a principal bundle $\pi : \mathcal{E} \longrightarrow \mathcal{B}$ with structure group $\mathcal{G}$.

In this framework, a configuration is an embedding of principal bundles of $\pi : \mathcal{E} \longrightarrow \mathcal{B}$ into the trivial bundle $\pi_0 : \mathbb{R}^n \times \mathcal{G} \longrightarrow \mathcal{B}$. A change of configuration is a deformation. The material response is supposed to depend on the 1-jet of the deformation. We introduce the notion of uniformity and isotropy group in terms of jets. If the body $\pi : \mathcal{E} \longrightarrow \mathcal{B}$ enjoys smooth uniformity we can characterize the homogeneity in terms of three connections: one linear connection Γ on $\mathcal{B}$ and two connections in the principal bundle $\pi : \mathcal{E} \longrightarrow \mathcal{B}$: Λ (which is defined from a global section $\mathcal{P} : \mathcal{B} \longrightarrow \mathcal{E}$) and $\bar{\Lambda}$. In fact, it is proved that $\mathcal{B}$ is locally homogeneous if and only if the torsion tensor

†This work has been partially supported through grants of the National Science and Engineering Research Council of Canada and DGICYT-Spain, Proyect PB94-0106.

T of Γ identically vanishes and the global section $\mathcal{P} : \mathcal{B} \longrightarrow \mathcal{E}$ is parallel with respect to $\bar{\Lambda}$.

These results recover the ones for second grade materials [3, 4, 12, 9] (see also [7, 8]), Cosserat continua [11, 6] and continua with vector microstructure [5].

Our approach may be considered as the natural generalization of the continuous theories of inhomogeneities of Noll [18] and Wang [20] (see also [19]). An alternative approach based in a defective crystalline lattice due to Kondo, Bilby and Kröner [14, 1, 15]) was recently updated by Kröner [16]. The use of principal bundles formalism in elastoplasticity theories may enjoy interesting features as the recent work by Epstein and Maugin shows [10] (see also [17, 16]).

2. Continua with microstructure. Uniformity and material symmetries

An n-dimensional body $\mathcal{B}$ is said to be a continuum with microstructure if there exists a bundle $\tilde{\pi} : \tilde{\mathcal{E}} \longrightarrow \mathcal{B}$ associated with some principal bundle $\pi : \mathcal{E} \longrightarrow \mathcal{B}$ with structure group $\mathcal{G}$. The standard fibre $\mathcal{F}$ of $\tilde{\mathcal{E}}$ is the manifold of parameters. We assume that $\mathcal{F}$ has dimension m and dim $\mathcal{B} = n \leq 3$.

Denote by $\pi_0 : \mathbb{R}^3 \times \mathcal{G} \longrightarrow \mathbb{R}^3$ the trivial bundle. Therefore, a configuration of $\mathcal{E}$ is a principal bundle embedding $\tilde{\Phi} : \mathcal{E} \longrightarrow \mathbb{R}^3 \times \mathcal{G}$ which induces the identity between the structure groups. We denote by $\Phi : \mathcal{B} \longrightarrow \mathbb{R}^3$ the induced embedding between the bases.

A deformation is a change of configuration, that is, given two configurations $\tilde{\Phi}_i : \mathcal{E} \longrightarrow \mathbb{R}^3 \times \mathcal{G}$, $i = 1, 2$, $\tilde{\kappa} = \tilde{\Phi}_2 \circ \tilde{\Phi}_1^{-1}$, which is a principal bundle isomorphism from $\tilde{\Phi}_1(\mathcal{E})$ into $\tilde{\Phi}_2(\mathcal{E})$ inducing the identity between the structure groups and covering the diffeomorphism $\kappa = \Phi_2 \circ \Phi_1^{-1} : \Phi_1(\mathcal{B}) \longrightarrow \Phi_2(\mathcal{B})$.

We assume that the material response is completely characterized by a scalar function which depends on the first derivative of the deformation. The constitutive equation is:

$$W = W(j^1_{\tilde{X}, \tilde{\kappa}(\tilde{X})}\ \tilde{\kappa}) \ . \tag{1}$$

We can consider equivalence classes of local principal bundle isomorphisms (as in the Appendix A) and then the constitutive equation more appropiately reads as follows:

$$W = W(j^1_{X, \kappa(X)}\ \tilde{\kappa}) \ , \tag{2}$$

where $j^1_{X,\kappa(X)}\ \tilde{\kappa}$ denotes the equivalence class of $j^1_{\tilde{X},\tilde{\kappa}(\tilde{X})}\ \tilde{\kappa}$.

From now on, we fix a reference configuration $\tilde{\Phi}_0$, and make the obvious identifications: $\mathcal{B} = \Phi_0(\mathcal{B})$, $\mathcal{E} = \tilde{\Phi}_0(\mathcal{E})$.

DEFINITION 1. A continuum with microstructure $\mathcal{B}$ is said to be uniform if for every pair of points $X, Y \in \mathcal{B}$ there exists a local isomorphism of principal bundles $\tilde{\Phi}$ (inducing the identity between the structure groups) such that $\Phi(X) = Y$ and

$$W(j^1_{\tilde{\Phi}(\tilde{X}),\tilde{\kappa}(\tilde{\Phi}(\tilde{X}))}\ \tilde{\kappa} \circ j^1_{\tilde{X},\tilde{\Phi}(\tilde{X})}\ \tilde{\Phi}) = W(j^1_{\tilde{\Phi}(\tilde{X}),\tilde{\kappa}(\tilde{\Phi}(\tilde{X}))}\ \tilde{\kappa})$$
$$, \forall \tilde{X} \in \pi^{-1}(X), \forall j^1_{\tilde{\Phi}(\tilde{X}),\tilde{\kappa}(\tilde{\Phi}(\tilde{X}))}\ \tilde{\kappa}\,, \quad (3)$$

where $\circ$ denotes the composition of jets.

With the obvious notations, the uniformity condition may be equivalently written as

$$W(j^1_{\Phi(X),\kappa(\Phi(X))}\ \tilde{\kappa} \circ j^1_{X,\Phi(X)}\ \tilde{\Phi}) = W(j^1_{\Phi(X),\kappa(\Phi(X))}\ \tilde{\kappa})\ ,\ \forall j^1_{\Phi(X),\kappa(\Phi(X))}\ \tilde{\kappa}\,, \quad (4)$$

where $\circ$ denotes the composition of equivalence classes of jets.

Such a 1-jet (and its class) will be called a local uniformity from X to Y.

A material symmetry at a point $X \in \mathcal{B}$ is a 1-jet $j^1_{\tilde{X},\tilde{\Phi}(\tilde{X})}\ \tilde{\Phi}$ where $\tilde{\Phi}$ is a local isomorphism of principal bundles (inducing the identity between the structure groups) such that $\pi(\tilde{\Phi}(\tilde{X})) = \pi(\tilde{X}) = X$ and

$$W(j^1_{\tilde{\Phi}(\tilde{X}),\tilde{\kappa}(\tilde{\Phi}(\tilde{X}))}\ \tilde{\kappa} \circ j^1_{\tilde{X},\tilde{\Phi}(\tilde{X})}\ \tilde{\Phi}) = W(j^1_{\tilde{\Phi}(\tilde{X}),\tilde{\kappa}(\tilde{\Phi}(\tilde{X}))}\ \tilde{\kappa})$$
$$, \forall \tilde{X} \in \pi^{-1}(X), \forall j^1_{\tilde{\Phi}(\tilde{X}),\tilde{\kappa}(\tilde{\Phi}(\tilde{X}))}\ \tilde{\kappa}\,. \quad (5)$$

Again, by using equivalence classes of jets, we can write Equation (5) as follows

$$W(j^1_{\Phi(X),\kappa(\Phi(X))}\ \tilde{\kappa} \circ j^1_{X,\Phi(X)}\ \tilde{\Phi}) = W(j^1_{\Phi(X),\kappa(\Phi(X))}\ \tilde{\kappa})\ , \quad \forall j^1_{\Phi(X),\kappa(\Phi(X))}\ \tilde{\kappa}\,. (6)$$

From (5) (or 6) we deduce that the collection $\bar{\mathcal{G}}(X)$ of all material symmetries at X forms a group which is called the isotropy group at X. Of course, the collection $\mathcal{G}(X)$ of all the induced 1-jets on the base $\mathcal{B}$ also forms a group.

3. Uniform continua with microstructure

Consider the family $\bar{\Omega}(\mathcal{B})$ of all the local uniformities $j^1_{X,\Phi(X)}\tilde{\Phi}$. (Here we use the notations introduced in Appendix A). We have $\bar{\Omega}(\mathcal{B}) \subset \tilde{J}^1(\mathcal{E})$.

Definition 2. A continuum with microstructure $\mathcal{B}$ is said to be smoothly uniform if $\bar{\Omega}(\mathcal{B})$ is a Lie subgroupoid which admits a smooth global section. Such a section it is called a global smooth uniformity.

From now on, we suppose that $\mathcal{B}$ enjoys global smooth uniformity and $\sigma : \mathcal{B} \times \mathcal{B} \longrightarrow \bar{\Omega}(\mathcal{B})$ is a global uniformity, that is, σ is a smooth global section of $(\bar{\alpha}, \bar{\beta}) : \bar{\Omega}(\mathcal{B}) \longrightarrow \mathcal{B} \times \mathcal{B}$.

Choose a point $X_0 \in \mathcal{B}$ and define $S : \mathcal{B} \longrightarrow \bar{\Omega}(\mathcal{B})$ by $S(X) = \sigma(X_0, X)$. Next, choose a non-holonomic frame $\bar{Z}_0 = j^1_{e_1,\tilde{\Psi}(e_1)}\tilde{\Psi}$ at X_0 (see Appendix C) and put:

$$\mathcal{S}(X) = \bar{S}(X) \circ \bar{Z}_0 \ , \ \forall X \in \mathcal{B} \ ,$$

where $\bar{S}(X)$ is the representative in $S(X)$ with source $\tilde{\Psi}(e_1)$. In other words, $\mathcal{S} : \mathcal{B} \longrightarrow \bar{F}\mathcal{E}$ is a non-holonomic parallelism on $\mathcal{B}$. Sometimes we shall refer to $\mathcal{S}$ as a field of uniformities.

Definition 1. A non-holonomic frame $\bar{Z}_0$ at X_0 will be called a reference crystal at that point.

By using a reference crystal $\bar{Z}_0$ we obtain a Lie subgroup $\bar{G}$ of $\bar{G}(n, \mathcal{G})$ as follows:

$$\bar{G} = \{\bar{Z}_0^{-1} \circ \bar{Z} \circ \bar{Z}_0 \mid Z \in \bar{\mathcal{G}}(X_0)\} \ , \tag{7}$$

where $\bar{Z}$ denotes the representative of the class Z with source $\tilde{\Psi}(e_1)$.

By applying the results of Appendix D, we know that $\mathcal{S}$ induces:

- A global section $\mathcal{P}$ of the principal bundle $\pi : \mathcal{E} \longrightarrow \mathcal{B}$ (and hence, a connection Λ in $\mathcal{E}$).
- A linear parallelism $\mathcal{Q}$ on $\mathcal{B}$ (and hence a linear connection Γ on $\mathcal{B}$).
- A connection $\bar{\Lambda}$ in $\mathcal{E}$.

Notice that there is a degree of freedon in the choice of $\mathcal{S}$. In fact, if we change the reference crystal $\bar{Z}_0$ to a new one $\bar{Z}'_0$, then $\bar{Z}'_0 = \bar{Z}_0(A, B, C)$, where $(A, B, C) \in \bar{G}(n, \mathcal{G})$. Thus, $\mathcal{S}' = \mathcal{S}(A, B, C)$ is a new field of uniformities.

Furthermore, there is another degree of freedom. Suppose that $\bar{G}$ is a continuous Lie subgroup of $\bar{G}(n, \mathcal{G})$. Hence we prolongate a non-holonomic parallelism $\mathcal{S}$ by $\bar{G}$ and obtain a $\bar{G}$-reduction of $\bar{F}\mathcal{E}$. Therefore, all the sections $\mathcal{S}(A(X), B(X), C(X))$, where $(A, B, C) : \mathcal{B} \longrightarrow \bar{G}$, are admissible non-holonomic parallelisms or, in other words, new fields of uniformities.

Remark 1. There is a more general class of continua with microstructure. Suppose that $\mathcal{B}$ only enjoys local smooth uniformity, that is, $\bar{\Omega}(\mathcal{B})$ is a Lie subgroupoid which only admits local sections (in other words, local uniformities). As above, we fix a point X_0 at $\mathcal{B}$ and a non-holonomic frame at X_0. Proceeding in the same way, we obtain local sections of $\bar{\pi} : \mathcal{B} \longrightarrow \bar{F}\mathcal{E}$ and, by prolongation, a $\bar{G}$-reduction. We call such a reduction a $\bar{G}$-structure.

4. Homogeneous continua with microstructure

Definition 3. We say that $\mathcal{B}$ is homogeneous if there exists a global configuration $\tilde{\kappa}$ such that:

1. $\kappa : \mathcal{B} \longrightarrow R^3$ is an embedding into R^n, i.e. $\kappa(\mathcal{B}) \subset R^n$; and
2. $\mathcal{S} = \tilde{\kappa}^{-1}$ is a uniformity field.

More precisely, for each $X \in \mathcal{B}$, let $\tilde{\mathcal{A}}_X : R^n \times \mathcal{G} \longrightarrow \mathcal{E}$ be the bundle isomorphism defined by

$$\tilde{\mathcal{A}}_X(r, R) = \tilde{\kappa}^{-1}(r + \kappa(X), R) . \tag{8}$$

Then $\mathcal{B}$ is homogeneous if $\mathcal{S}(X) = j^1_{e_1, \tilde{\mathcal{A}}_X(e_1)} \tilde{\mathcal{A}}_X$ is a uniformity field. The continuum $\mathcal{B}$ is said to be locally homogeneous if every point of $\mathcal{B}$ has a neighborhood which is homogeneous.

In that case, there exist local coordinates (x^i) in R^n such that

$$\mathcal{S}(x^i) = \left(x^i, \mathcal{P}^\alpha(x), 1, \frac{\partial \mathcal{P}^\alpha}{\partial x^j}\right) . \tag{9}$$

That is, $\mathcal{S}$ is an integrable prolongation.

Conversely, let $\mathcal{S}$ be a uniformity field for $\mathcal{B}$. If $\mathcal{S}$ is an integrable prolongation, then $\mathcal{B}$ is locally homogeneous.

Thus, we deduce the following result which characterizes geometrically the homogeneity of a medium with structure.

Theorem 1. *A continuum with microstructure $\mathcal{B}$ is locally homogeneous if and only if it admits a field of uniformities which is an integrable prolongation.*

In order to decide if a continuum $\mathcal{B}$ is locally homogeneous, we proceed as follows. Suppose first that there are no material symmetries except the identity, i.e., $\bar{G} = (e, 1, 0)$. Take a field of uniformities $\mathcal{S}$, with associated connections Γ, Λ, and $\bar{\Lambda}$. Compute the torsion tensor of the linear connection Γ. If it vanishes, we then check if the global section $\mathcal{P}$ is parallel with respect to $\bar{\Lambda}$. If it is not, we change to another field of uniformities $\mathcal{S}' = \mathcal{S}(A, B, C)$ by means of a change of reference crystal and consider the new three connections Γ', Λ', and $\bar{\Lambda}'$. Clearly, $\Gamma' = \Gamma$, and $\mathcal{P}A$ is parallel with respect to $\bar{\Lambda}'$ if and only if $\mathcal{P}$ is so also. If we can choose $(A, B, C) \in \bar{G}(n, \mathcal{G})$ such that $\mathcal{P}$ is parallel with respect to $\bar{\Lambda}'$, we have finished, and $\mathcal{B}$ is locally homogeneous.

Now, suppose that $\bar{G}$ is not trivial. In this case, we have many choices for a uniformity field. The geometrical answer for a local homogeneity characterization needs to develop an appropiate study of the integrability problem for $\bar{G}$-structures.

A. Lie groupoid associated with a principal bundle

Let $\pi : \mathcal{E} \longrightarrow \mathcal{B}$ be a principal bundle with structure group $\mathcal{G}$. Denote by $J^1(\mathcal{E})$ the manifold of 1-jets $j^1_{\tilde{X},\tilde{\Phi}(\tilde{X})}\tilde{\Phi}$ of local automorphisms $\tilde{\Phi}$ of $\mathcal{E}$ such that $\tilde{\Phi}(\tilde{Y}A) = \tilde{\Phi}(\tilde{Y})A$, $\forall \tilde{Y} \in \mathcal{E}, \forall A \in \mathcal{G}$. Notice that $J^1(\mathcal{E}) \subset \Pi^1(\mathcal{E}, \mathcal{E})$, the Lie groupoid of the invertible 1-jets of the manifold $\mathcal{E}$. We define an equivalence relation on $J^1(\mathcal{E})$ as follows: $j^1_{\tilde{X},\tilde{\Phi}(\tilde{X})}\tilde{\Phi} \sim j^1_{\tilde{X}A,\tilde{\Phi}(\tilde{X})A}\tilde{\Phi}$. The equivalence class of $j^1_{\tilde{X},\tilde{\Phi}(\tilde{X})}\tilde{\Phi}$ will be denoted by $j^1_{X,\Phi(X)}\tilde{\Phi}$, where $X = \pi(\tilde{X})$ and Φ is the induced diffeomorphism between the bases. Denote by $\tilde{J}^1(\mathcal{E})$ the quotient space $J^1(\mathcal{E})/\mathcal{G}$. If we define

$$\bar{\alpha}([j^1_{X,\Phi(X)}\tilde{\Phi}]) = X \ , \ \bar{\beta}([j^1_{X,\Phi(X)}\tilde{\Phi}]) = \Phi(X) \ ,$$

we can easily check that $\tilde{J}^1(\mathcal{E})$ is a Lie groupoid over $\mathcal{B}$ with source and target maps $\bar{\alpha}, \bar{\beta} : \tilde{J}^1(\mathcal{E}) \longrightarrow \mathcal{B}$.

Furthemore, the set of induced 1-jets $j^1_{X,\Phi(X)}\Phi$ is just $\Pi^1(\mathcal{B}, \mathcal{B})$.

B. Bundles associated with principal bundles

Let $\pi : \mathcal{E} \longrightarrow \mathcal{B}$ be a principal bundle with structure group $\mathcal{G}$. Suppose that $\mathcal{G}$ acts on the left on a manifold $\mathcal{F}$, namely $\mathcal{G} \times \mathcal{F} \longrightarrow \mathcal{F}$. We define on the product manifold $\mathcal{E} \times \mathcal{F}$ the following action of $\mathcal{G}$:

$$\begin{aligned} (\mathcal{E} \times \mathcal{F}) \times \mathcal{G} &\longrightarrow \mathcal{E} \times \mathcal{F} , \\ (\tilde{X}, \xi) A &\rightsquigarrow (\tilde{X} A, A^{-1} \xi) . \end{aligned} \tag{10}$$

Denote by $\tilde{\mathcal{E}} = \dfrac{\mathcal{E} \times \mathcal{F}}{\mathcal{G}}$ the quotient space and by $\tilde{\pi} : \tilde{\mathcal{E}} \longrightarrow \mathcal{B}$ the canonical projection. We have that $\tilde{\pi} : \tilde{\mathcal{E}} \longrightarrow \mathcal{B}$ is a fibre bundle with standard fibre $\mathcal{F}$ which is called an associated fibre bundle with $\mathcal{E}$.

C. Non-holonomic frames of a principal bundle

Let $\pi : \mathcal{E} \longrightarrow \mathcal{B}$ be a principal bundle with projection π and structure group $\mathcal{G}$. Consider the trivial principal bundle $R^n \times \mathcal{G} \longrightarrow \mathbb{R}^n$, where dim $\mathcal{B} = n$. Denote by e_1 the element $e_1 = (0, e)$, where e is the neutral element of $\mathcal{G}$.

A non-holonomic frame of $\mathcal{E}$ at a point $X \in \mathcal{B}$ is a 1-jet $j^1_{e_1, \tilde{\Phi}(e_1)}\ \tilde{\Phi}$ of a local principal bundle isomorphism $\tilde{\Phi} : \mathbb{R}^n \times \mathcal{G} \longrightarrow \mathcal{E}$, where $\tilde{\Phi}$ induces the identity between the structure groups, and $\pi(\tilde{\Phi}(e_1)) = X$. The collection of all non-holonomic frames at all the points of $\mathcal{B}$ is denoted by $\bar{F}\mathcal{E}$ and we have $\bar{F}\mathcal{E} \subset F(\mathcal{E})$, where $F(\mathcal{E})$ denotes the linear frame bundle of the manifold $\mathcal{E}$.

Take canonical coordinates (r^i), $1 \leq i, j, k, \cdots \leq n$, on $\mathbb{R}^n$ and coordinates (R^α), $1 \leq \alpha, \beta, \gamma, \cdots \leq$ dim $\mathcal{G}$, on $\mathcal{G}$ (we can choose normal coordinates on $\mathcal{G}$, for instance). On $\mathcal{E}$ we have fibred coordinates (x^i, X^α). We have

$$\tilde{\Phi}(r, R) = (\Phi(r), \varphi(r) R) , \tag{11}$$

where $\Phi : \mathbb{R}^n \longrightarrow \mathcal{B}$ and $\varphi : \mathbb{R}^n \longrightarrow \mathcal{G}$. We get

$$j^1_{e_1, \tilde{\Phi}(e_1)}\ \tilde{\Phi} = \left(\Phi^i(0), \varphi^\alpha(0), \frac{\partial \Phi^i}{\partial r^j}(0), 0, \frac{\partial \varphi^\alpha}{\partial r^j}(0), \varphi^\alpha(0) \right) . \tag{12}$$

We have used the following local coordinates:

$$\begin{aligned} \mathcal{B} &: (x^i) , \\ \mathcal{E} &: (x^i, X^\alpha) , \\ F(\mathcal{E}) &: (x^i, X^\alpha; x^i{}_{,j}, x^i{}_{,\beta}, X^\alpha{}_{,j}, X^\alpha{}_{,\beta}) . \end{aligned} \tag{13}$$

With these notations the coordinates of $j^1_{e_1, \tilde{\Phi}(e_1)}\ \tilde{\Phi}$ are $(x^i, X^\alpha; x^i{}_{,j}, X^\alpha{}_{,j})$. We deduce that $\bar{F}\mathcal{E}$ is a $(n + \text{dim } \mathcal{G})(n + 1)$-dimensional submanifold of $F(\mathcal{E})$.

Furthermore, if we consider the elements $j^1_{e_1,\tilde{\Phi}(e_1)}\ \tilde{\Phi}$ from $R^n \times \mathcal{G}$ into itself such that $\Phi(0) = 0$, we obtain a Lie group denoted by $\bar{G}(n, \mathcal{G})$ whose elements are of the form (A, B, C), where $A \in \mathcal{G}$, $B \in Gl(n, \mathbb{R})$ and $C \in \text{Lin}\ (\mathbb{R}^n, \mathfrak{g})$, $\mathfrak{g}$ being the Lie algebra of $\mathcal{G}$. Thus, $\bar{G}(n, \mathcal{G})$ may be identified with the product $\mathcal{G} \times Gl(n, \mathbb{R}) \times \text{Lin}\ (\mathbb{R}^n, \mathfrak{g})$, the multiplication law given by the following formula obtained by applying the chain rule:

$$(A_1, B_1, C_1)(A_2, B_2, C_2) = (A = A_1A_2, B = B_1B_2, C = A_2C_1B_2 + A_1C_2)\ , \tag{14}$$

with the following definitions:

- If $A \in \mathcal{G}$ and $C \in \text{Lin}\ (\mathbb{R}^n, \mathfrak{g})$, then AC is the composition $\mathbb{R}^n \xrightarrow{C} \mathfrak{g} \xrightarrow{A} \mathfrak{g}$, the second mapping being the induced one from the right translation by A.
- If $B \in Gl(n, \mathbb{R})$ and $C \in \text{Lin}\ (\mathbb{R}^n, \mathfrak{g})$, then CB is the composition $\mathbb{R}^n \xrightarrow{B} \mathbb{R}^n \xrightarrow{C} \mathfrak{g}$.

A simple computation shows that $\bar{\pi} : \bar{F}\mathcal{E} \longrightarrow \mathcal{B}$, where $\bar{\pi}$ is the canonical projection, is a principal bundle over $\mathcal{B}$ and with structure group $\bar{G}(n, \mathcal{G})$. $\bar{F}\mathcal{E}$ will be called the non-holonomic frame bundle of $\mathcal{E}$. We denote by $\rho : \bar{F}\mathcal{B} \longrightarrow F\mathcal{B}$ and $\theta : \bar{F}\mathcal{B} \longrightarrow \mathcal{E}$ the canonical projections.

D. Non-holonomic parallelisms

Definition 4. A global section $\mathcal{S} : \mathcal{B} \longrightarrow \bar{F}\mathcal{E}$ is called a non-holonomic parallelism on $\mathcal{E}$.

By using the projections ρ and θ, $\mathcal{S}$ determines:

- A global section $\mathcal{P} : \mathcal{B} \longrightarrow \mathcal{E}$;
- A linear parallelism $\mathcal{Q}$ on $\mathcal{B}$;
- A connection $\bar{\Lambda}$ in $\pi : \mathcal{E} \longrightarrow \mathcal{B}$, by defining the horizontal subspaces as follows. Let $\mathcal{S}(X) = j^1_{e_1,\tilde{\Phi}(e_1)}\tilde{\Phi}$ be such that $\tilde{\Phi}(r, R) = \varphi(r)R$, where $\varphi(r) = \tilde{\Phi}(r, e)$. We define a horizontal subspace $H_{\mathcal{P}(X)} = d\varphi(0)(T_0\mathbb{R}^n)$ and, then we transport $H_{\mathcal{P}(X)}$ by the action of $\mathcal{G}$.

Remark 2. Roughly speaking, a nonoholonomic frame at a point X is an infinitesimal element of connection, that is, a horizontal subspace over X.

Conversely, let $\mathcal{P}$ be a global section of $\mathcal{E}$ and $\mathcal{Q}$ a linear parallelism on $\mathcal{B}$. We obtain a non-holonomic parallelism $\mathcal{P}^1(\mathcal{Q})$ on $\mathcal{E}$ by defining $\mathcal{S}(X)$ to be the "linear connection" at $\mathcal{P}(X)$ given by the horizontal subspace spanned by the tangent vectors $\{d\mathcal{P}(X)(\mathcal{Q}(X))\}$.

DEFINITION 5. (1) A non-holonomic parallelism $\mathcal{S}$ is called a prolongation if $\mathcal{S} = \mathcal{P}^1(\mathcal{Q})$. (2) $\mathcal{S}$ is called an integrable prolongation if $\mathcal{S} = \mathcal{P}^1(\mathcal{Q})$ and $\mathcal{Q}$ is integrable.

Suppose that $\mathcal{P}(x^i) = (x^i, \mathcal{P}^\alpha(x))$ and $\mathcal{Q}(x^i) = (x^i, \mathcal{Q}^i_j(x))$. Hence

$$\mathcal{P}^1(\mathcal{Q})(x^i) = \left(x^i, \mathcal{P}^\alpha, \mathcal{Q}^i_j, \mathcal{Q}^k_j \frac{\partial \mathcal{P}^\alpha}{\partial x^k}\right) .$$

Therefore $\mathcal{S}(x^i) = (x^i, \mathcal{P}^\alpha, \mathcal{Q}^i_j, \mathcal{R}^\alpha_j)$ is a prolongation if and only if

$$\mathcal{R}^\alpha_j = \mathcal{Q}^k_j \frac{\partial \mathcal{P}^\alpha}{\partial x^k} .$$

and, $\mathcal{S}$ is an integrable prolongation if and only if there exist local coordinates (x^i) on $\mathcal{B}$ such that

$$\mathcal{Q}^i_j = \delta^i_j , \qquad (15)$$

$$\mathcal{R}^\alpha_j = \frac{\partial \mathcal{P}^\alpha}{\partial x^j} . \qquad (16)$$

If $\mathcal{S}$ is a non-holonomic parallelism on $\mathcal{E}$ then it defines three connections:
- A linear connection Γ on $\mathcal{B}$ induced by the linear parallelism $\mathcal{Q}$ and with Christoffel components:

$$\Gamma^i_{jk} = -(\mathcal{Q}^{-1})^l_k \frac{\partial \mathcal{Q}^i_l}{\partial x^j} . \qquad (17)$$

- A connection Λ in the principal bundle $\pi : \mathcal{E} \longrightarrow \mathcal{B}$ whose horizontal subspace at $\mathcal{P}(X)$ is obtained by transporting the tangent space $T_X\mathcal{B}$. Then we transport it by the action of the Lie group $\mathcal{G}$. The horizontal subspaces along $\mathcal{P}$ are locally spanned by

$$\left\{\frac{\partial}{\partial x^i} + \frac{\partial \mathcal{P}^\alpha}{\partial x^i}\frac{\partial}{\partial X^\alpha}\right\} . \qquad (18)$$

- A connection $\bar{\Lambda}$ in the principal bundle $\pi : \mathcal{E} \longrightarrow \mathcal{B}$ whose horizontal subspaces along $\mathcal{P}$ are locally spanned by

$$\left\{\mathcal{Q}^j_i \frac{\partial}{\partial x^j} + \mathcal{R}^\alpha_i \frac{\partial}{\partial X^\alpha}\right\} . \qquad (19)$$

From (17), (18) and (19) we deduce the following.

THEOREM 2. *A non-holonomic parallelism $\mathcal{S}$ is an integrable prolongation if and only if Γ is symmetric and $\Lambda = \bar{\Lambda}$.*

Theorem 2 may be rephrased as follows. Denote by T the torsion tensor of Γ. Hence we have.

THEOREM 3. *A non-holonomic parallelism $\mathcal{S}$ is an integrable prolongation if and only if T identically vanishes and $\mathcal{P}$ is parallel with respect to the connection $\bar{\Lambda}$.*

The result follows taking into account that Λ and $\bar{\Lambda}$ coincide if and only if

$$d\mathcal{P}(X)(Q_i) = (Q_i(X))^{\bar{H}} \ , \ \forall X \in \mathcal{B} \ , \ 1 \leq i \leq n \ ,$$

where $\{Q_1, \dots, Q_n\}$ is the linear parallelism defined by $\mathcal{Q}$ and $U^{\bar{H}}$ denotes the horizontal lift of a tangent vector $U \in T_X\mathcal{B}$ to $\mathcal{E}$.

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