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On Cartan's Differential Geometry of Torque Stress in Cylindrical Solids

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1. Introduction

The Cartan's interpretation of torque stress in stress space of continuum mechanics compute the torque stress and torsion components of a twisted long cylinder which is equivalent to a screw dislocated cylinder. This result is in agreement with previous results obtained by Eshelby without involving non-Riemanniam geometry.

In 1922 Elie Cartan [4] laid the foundations of the aplication of non Riemannian Geometry to continuum mechanics on a paper entitled : "Sur une generalisation de la notion de Courboure de Riemann et spaces a Torsion". In this paper Cartan present two important ideas concerning torsion of space and continuum mechanics. The first one was connected to the fact that couples (torques) would be associeted to the idea of torsion and the other that the Cosserats [5] conditions of equilibrium of continua would lead to the vanishing of the couple and consequently to the symmetry of stress tensor. Later Kondo [9], Bilby[3] and Kröner [10] applied Cartan's idea to dislocated crystals, In this note We shall be concern with the Minagawa - Amari [14, 1, 2] idea of the Dual non-Riemannian Geometry to compute the dual torsion of a twisted cylinder via the torque stress-torsion link. As pointed out recently by Kröner [11]: " ... Riemann-Cartan geometry describes dislocations in the form of Cartan's torsion of the strain-space ... and the specific response to dislocations in torque stress which arises as Cartan's torsion of stress space ...". Kröner remark guides us on the application of the non-Riemannian geometry to the investigation of a twisted cylinder. Also as pointed out by Eshelby [6] the twisted of a cylinder leads to screw dislocation which agrees with our computation here. In a dualistic non-Riemannian theory of stresses their authors

[14, 1] work with a 3^{rd} rank couple stress μ_{ijk}, here. However we make use of the 2^{nd} rank torque stress tensor τ_{ij}, both of them are related by the expression $\tau_{ij} = \epsilon_{jkl}\mu_{ikl}$ nevertheless with aproaches are equivalent.

2. The dual stress space and Cartan's Torsion of a Twisted Cylinder

Let us consider the Cosserats [5] brother's formula for the equilibrium state

$$\nabla.\sigma = 0 \tag{1}$$

$$\nabla.\tau + \vec{\sigma} = 0, \tag{2}$$

where σ is the stress tensor $\sigma = \sigma_{ij}$ and $\vec{\sigma} = (\sigma_i = \epsilon_{ijk}\sigma_{jk})$ is the skew-symetric part of the torsion tensor [12], and $\tau = (\tau_{ij})$ is the torque stress. In fact formula (2) can be writen in component form as

$$\partial_i\tau_{ij} + \epsilon_{jkl}\sigma_{kl} = 0, \tag{3}$$

which is equivalent to Amari's formula

$$\partial_i\mu_{ijk} + \sigma_{[jk]} = 0, \tag{4}$$

where μ_{ijk} is couple-stress of space and [12] $\tau_{ij} = \epsilon_{ipq}\mu_{pqj}$. Formula (4) or (3) agrees with Cartan's hypothesis (ii) since from the Einstein's equations of continuum mechanics

$$G_{[ij]} = \sigma_{[ij]} = \partial_k\mu_{kij}, \tag{5}$$

where $G_{[ij]} = \frac{1}{4}\epsilon^{rst}\epsilon^{kij}R_{[rst]k}$ is the skew-symmetric part of the Einstein tensor. One can express the couple stress as

$$\mu_{plk} = \frac{1}{2}\epsilon^{prs}\epsilon^{tlk}\widetilde{S}_{rst}, \tag{6}$$

by using the identity [14]

$$\partial_{[i}\widetilde{S}_{jk]m} = \frac{1}{2}\widetilde{R}_{[ijk]m}. \tag{7}$$

Substituition of second equality in (5) into (3) yields

$$\partial_i(\tau_{ij} - \frac{1}{2}\epsilon^{irs}\widetilde{S}_{rsj}) = 0. \tag{8}$$

A particular solution of the differential relation (8) is

$$\tau_{ij} = \frac{1}{2}\epsilon^{irs}\widetilde{S}_{rsj} + \text{const} = \frac{1}{2}\alpha_{ij}, \tag{9}$$

where α_{ij} represents the dislocation density in the continuum theory of defects [10].

In linear approximation it is possible to consider a relation between Nye's curvature K_{kl} and torque stress of the form [12]

$$\tau_{ij} = A_{ijkl}K_{kl}, \tag{10}$$

where $A_{ijkl} = a_1\delta_{ij}\delta_{kl} + a_2\delta_{ik}\delta jl + a_3\delta_{il}\delta_{jk}$. For the choice of constants $a_1 = 1, a_2 = 1, a_3 = -1$ one obtains the Nye's relation [15]

$$\tau_{ij} = -K_{(ij)} + \frac{1}{2}\tau_{ij}K_{kl} = \frac{1}{2}\alpha_{ij}, \tag{11}$$

which agrees with relation (9). For this choice of the $a_i^{\prime}s$ constants one obtains

$$\widetilde{S}_{pqj} = 2\epsilon_{ipq}(\partial_{[i}w_{j]} + \frac{1}{2}\nabla.\vec{\omega}) + \text{const}, \tag{12}$$

since $K_{kl} = \partial_k w_l$ and $w_l = \epsilon_{lkr}\omega_{kr} = \epsilon_{lkr}\partial_{[r}U_{k]}$, where U_k is the strain vector and $(\omega_l = \vec{\omega})$ is the rotation vector. Since $\vec{\omega} = \nabla x\vec{u}$, $\nabla.\vec{\omega} = 0$ and (12) reduces to

$$\widetilde{S}_{pqj} = 2\epsilon_{ipq}\partial_{[i}\omega_{j]} + const. \tag{13}$$

Let us now compute the dual torsion of a twisted cylinder where no external forces act and a torque stress diferent from zero. In this example as whe shall notice the stress tensors are symmetric and formula (3) reduces to

$$\partial_i\tau_{ij} = 0, \tag{14}$$

which has as a particular solution $\tau_{ij} = \text{const}$. From expression (9) one may conclude that the dual torsion is constant. Another way of showing that, although not o straightforward is to consider formula (13) and compute the rotation vector of the twisted cylinder from the strain vector of the Rod [13] we have

$$u_x = -\tau_{yz}, u_y = \tau_{xz}, u_z = \tau\psi(\text{x,y}), \tag{15}$$

where τ is the twist angle and ψ(x,y) is the torsion function. Computation of the rotation vector yields

$$\omega_x = \tau(\partial_y\psi - x), \tag{16}$$

$$\omega_y = -\tau(\partial_x \psi + y),$$

$$\omega_z = \tau z.$$

Substitution of (16) into (13) yields

$$\widetilde{S}_{232} = \widetilde{S}_{131} = -[\tau \nabla^2 \psi] + \text{const.} \tag{17}$$

The other components of the dual torsion vanish identically. The proof of the constancy of the dual torsion comes from the fact that the first Cosserat equation $\partial_i \sigma_{ij} = 0$ applied to the twist function of the Rod yields

$$\nabla^2 \psi = 0. \tag{18}$$

Of course substitution of (18) into (17) yields $\widetilde{S} = \text{const.}$. Note that if we choose $\psi = \tan^{-1}(\frac{y}{x})$ as a solution for (18) we have $\nabla^2 U_z = 0$ and our solution is very similar to the screw dislocated cylinder [8]. Computation of the dual torsion in the case of dislocated cylinder reads

$$\widetilde{S}_{xyz} = \frac{\mu b}{2\pi}. \tag{19}$$

If one takes into account Eshelby's [6] argument that the twist of a cylinder produces screw dislocations and vice-versa then $\widetilde{S}_{disloc} = \widetilde{S}_{twist}$ from (17). Since the dislocation density $\alpha = \frac{b}{\pi R^2}$ and $b = \frac{2\widetilde{S}\pi}{\mu}$ from (17), one has

$$\alpha = \frac{\widetilde{S}_{twist}}{\frac{1}{2}\mu R^2}. \tag{20}$$

But from the non-Riemannian theory of stresses the torque stress

$$\frac{M}{\pi R^2} = \widetilde{S}_{twist}, \tag{21}$$

where M is the external couple to the rod. Therefore from (21) and (20) one obtains

$$\alpha = \frac{M}{\frac{1}{2}\mu R^4 \pi}. \tag{22}$$

Expression (22) is the Eshelby's [6] formula for the twist per lenght of a dislocated rod. Therefore, making use of the non-Riemannian theory of dual stress and strains it is possible to obtain an expression for the twist of a rod undergoing plastic deformation. An interesting example of how to construct a rod with curvature and torsion has been given by Eshelby [6] which consists of marking out a thin straight rod in the material and cut it out. This rod is dotted with curvature and torsion due to the relaxatioon of internal stress. Other examples of torque stresses in materials has been given by E.Kröner in the linear approximation where equation (10) is valid [12]. [1]

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REFERENCES

[1] AMARI, S., Theory of deformations and stresses of ferromagnetic substances by finsler geometry, RAAG Memoirs, Vol. IV, Division D, (1968), 153–214.

[2] AMARI, S., Dualistic theory of non-Riemannian material manifolds, *Int. J. Engng. Sci.,* **19,12** (1981), 1581–1594.

[3] BILBY, B.A., BULHOUGH, R., SMITH, E., Continuous distributions of dislocations: a new application of the methods of non-Riemannian geometry, *Proc. Roy. Soc. London,* **231**A (1955), 263–273.

[4] CARTAN, E., Sur une generalisation de la notion de courboure de Riemann a espaces a torsion, *Comptes Rendus Ac. Sc. Paris,* (1922), 27 Feb 174, 593–595.

[5] COSSERAT, E., COSSERAT, F., Théorie des corps deformables, in "Traité de Physique", Ed. O.D. Chwolson, 2nd. Ed., Paris, 1909, 953–1173.

[6] ESHELBY, J.D., The force of an elastic singularity, *Phil. Trans. A,* **244** (1951), 87–144.

[7] HEHL, F.W., KRÖNER, E., Zum materialgesetz eines elastichen mediums mit momentenspannungen, *Zeit. für Naturforschung*, Band **200**, Helf 3, (1965), 336–340.

[8] JAGANNADHAM, K., MARCINKOWSKI, M.J., "The Unified Theory of Fracture", 1984.

[9] KONDO, K., On the geometrical and physical foundation of the theory of yielding in "Proc. 2^{nd}. Nat. Jap. Cong. Appl. Mech.", 1952, 41–47.

[1] An extension of equation (8) for the torque stress has been worked out by Hehl and Kröner [7].

[10] KRÖNER, E., Continuum theory of Defects, (1980) in school on the physics of defects (1979), Les Houches School of Physics, R. Balian et al. eds., Amsterdam, North–Holland, 1981, 219 – 315.

[11] KRÖNER, E., The mechanical state of defects in solids, *Int. J. Sol. Structures,* **29** (1992), 1849 – 1853.

[12] KRÖNER, E., The physical reality of torque stresses in continuum mechanics, *Int J. Engng. Sci.,* **1** (1963), 261 – 278.

[13] LANDAU, L., LIFCHITZ, I.M., "Theory of elasticity, Course in Theoretical Physics", vol. 8, Pergamon Press, 1986.

[14] MINAGAWA, S., Stress functions and stress-function spaces of non-metric connection for 3-dimensional micropolar elastostatics, *Int. J. Engng. Sci.,* **19**(12), (1981), 1705 – 1709.

[15] NYE, J., Some geometrical relations in dislocated crystals, *Acta Met.,* **1** (1953), 153 – 162.