EXTRACTA MATHEMATICAE Vol. **11**, Núm. 1, 37 – 40 (1996)

Exact Integration of the Cauchy-Green Tensor

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(to J.C. Simó, in memoriam)

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AMS *Subject Class.* (1991): 73CXX, 53B20, 35QXX

Given a smooth, positive definite symmetric two-tensor $\mathbf{C}$ defined on a connected and simply connected plane material body, a set of explicit formulae are given for the displacement field that makes $\mathbf{C}$ the *right Cauchy-Green strain tensor* of a regular configuration, provided $\mathbf{C}$ satisfies a necessary compatibility condition.

1. Introduction

Lets call *body*, or *material body* (see [1]) a -possibly unbounded- region $B \subset \mathbb{R}^n$ with the topology induced from the Euclidean one of the ambient space. Such B is sometimes referred to as *reference configuration*, and its points are called *material points*. In order to get the *current configuration*, points of B are mapped via a point-function, φ, called *deformation*

$$\varphi : B \longmapsto \mathbb{R}^n, P \longmapsto \varphi(P) = p$$

or, equivalently, by a *displacement vector field*

$$\mathbf{u} : B \longmapsto \mathbb{R}^n, \mathbf{u}(P) = \varphi(P) - P\,.$$

The vector $\mathbf{u}(P)$ represents the displacement of point P; the tensor function $\mathbf{F} = \nabla\varphi$ (i.e. tangent of φ) is called *deformation gradient*, and if $\det(\mathbf{F}) > 0$, φ is called a *regular configuration*. In terms of $\mathbf{F}$ the so called *right and left Cauchy-Green strain tensors* $\mathbf{C}$ and $\mathbf{B}$ are respectively defined by (see [1]):

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} \qquad \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$$

and, as

$$\mathbf{F} = \mathbf{I} + \nabla\mathbf{u}\,,$$

equivalently by

$$\mathbf{C} = \mathbf{I} + \nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}} + \nabla\mathbf{u}^{\mathrm{T}}\nabla\mathbf{u}$$
$$\mathbf{B} = \mathbf{I} + \nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}} + \nabla\mathbf{u}\nabla\mathbf{u}^{\mathrm{T}}$$

Obviously, $\mathbf{C}$ and $\mathbf{B}$ are symmetric tensors and also positive definite if det ($\mathbf{F}$) > 0. It is easy to show (see [2]) that the Riemann-Christoffel curvature tensor, $\mathbf{K}$, obtained by using the *right Cauchy-Green strain tensor* of a regular configuration, $\mathbf{C}$, as a metric tensor, satisfies

$$\mathbf{K} = 0 \qquad (*)$$

Once coordinates on $\mathbb{R}^n$ are selected, equations (*) are sometimes called *compatibility conditions* as they restrict the motion φ of a *body* in terms of its *deformation gradient*. There is a related question of some interest (see [2]); this is, given a tensor $\mathbf{C}$ that is symmetric and positive definite, when is $\mathbf{C}$ the right Cauchy-Green strain tensor of a *regular configuration*?.In case that only a local answer is seeked, the previous question has a positive answer (see [2]), provided condition (*) is satisfied. It is not known to this author whether a global answer to the above question is available in the literature.[1] In this paper we present an *explicit* answer to the global question under the *restrictive* dimensional assumption n=2; this solution is amenable to be implemented numerically.

2. Statement of Results

We will assume all along that an orthogonal set of coordinates (X, Y) is defined on $\mathbb{R}^2$, in such a way that

$$\mathbf{C} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

and the functions E, F, G are smooth enough so that the algebraic manipulations are valid. In terms of $\mathbf{u} = (u, v) : B \longmapsto \mathbb{R}^2$, the defining equation for

[1]When this paper was already in press, the author received notice of the work by R. T. Shield: SIAM J.Applied.Math. Vol 25, 3, Nov. 1973, where similar but not as explicit results to the ones in this paper are provided. The author wishes to thank Prof. G. P. Parry for providing him with that reference.

tensor $\mathbf{C}$ is equivalent to the first order coupled system

$$\begin{aligned} E &= 1+2\frac{\partial u}{\partial X}+\left(\frac{\partial u}{\partial X}\right)^2+\left(\frac{\partial v}{\partial X}\right)^2 \\ F &= \frac{\partial u}{\partial Y}+\frac{\partial v}{\partial X}+\frac{\partial u}{\partial X}\frac{\partial u}{\partial Y}+\frac{\partial v}{\partial X}\frac{\partial v}{\partial Y} \\ G &= 1+2\frac{\partial v}{\partial Y}+\left(\frac{\partial u}{\partial Y}\right)^2+\left(\frac{\partial v}{\partial Y}\right)^2 \end{aligned} \tag{1}$$

Furthermore, we observe that in case n=2, condition (*) is reduced to vanishing of the Gauss curvature of $\mathbf{C}$ (see [3]), $K_{\mathbf{C}}$,

$$\begin{aligned} K_{\mathbf{C}} = \frac{1}{2H}\Bigg[&\frac{\partial}{\partial X}\left(\frac{F}{EH}\frac{\partial E}{\partial Y}-\frac{1}{H}\frac{\partial G}{\partial X}\right) \\ &+\frac{\partial}{\partial Y}\left(\frac{2}{H}\frac{\partial F}{\partial X}-\frac{1}{H}\frac{\partial E}{\partial Y}-\frac{F}{EH}\frac{\partial E}{\partial X}\right)\Bigg] \end{aligned} \tag{2}$$

where $H=\sqrt{EG-F^2}$.

In this paper we show:

THEOREM. *Let $B\subset\mathbb{R}^2$ be an open connected and simply connected body, and E, F, $G: B\to\mathbb{R}^2$, $E>0$, $EG-F^2>0$ smooth functions satisfying the compatibility condition $K_C=0$ (2). Then there exists a displacement field $\mathbf{u}=(u,v): B\longmapsto\mathbb{R}^2$, solution of* (1), *explicitly obtained by quadratures.*

The solution of (1) is obtained in the following way:

Let $P_0, P\in B, P_0=(X_0,Y_0)$ be fixed, $P=(X,Y)$ arbitrary and $\gamma_{P_0}^P: I=[0,1]\mapsto B$ any smooth oriented path joining P_0 to P; then, the displacement of P in terms of the displacement of P_0 is obtained in two steps:

Step 1. Define μ (up to an additive constant depending on P_0) by the line integral:

$$\begin{aligned} \mu(P)=\mu(P_0)&+\int_{\gamma_{P_0}^P}\left(\frac{1}{H}\frac{\partial F}{\partial X}-\frac{F}{2EH}\frac{\partial E}{\partial X}-\frac{1}{2H}\frac{\partial E}{\partial Y}\right)dX \\ &+\int_{\gamma_{P_0}^P}\left(\frac{1}{2H}\frac{\partial G}{\partial X}-\frac{F}{2EH}\frac{\partial E}{\partial Y}\right)dY \end{aligned} \tag{3}$$

Step 2. Define $\mathbf{u}=(u,v)$ with components respectively given by the line integrals:

$$u(P) = u(P_0) + \int_{\gamma_{P_0}^P} \left(\sqrt{E}\cos\mu - 1\right) dX + \int_{\gamma_{P_0}^P} \left(\frac{F}{\sqrt{E}}\cos\mu - \frac{H}{\sqrt{E}}\sin\mu\right) dY \tag{4}$$

$$v(P) = v(P_0) + \int_{\gamma_{P_0}^P} \left(\sqrt{E}\sin\mu\right) dX + \int_{\gamma_{P_0}^P} \left(\frac{H}{\sqrt{E}}\cos\mu + \frac{F}{\sqrt{E}}\sin\mu - 1\right) dY \tag{5}$$

Remark. The well-definiteness of integrals (3), (4), (5) is consecuence of the compatibility condition.

Proof. By direct substitution. ∎

ACKNOWLEDGEMENTS

The author wishes to thank Prof. M. Epstein for helpful comments.

REFERENCES

[1] GURTIN, M.E., "An Introduction to Continuum Mechanics", Academic Press, New York, 1981.

[2] MARSDEN, J.E., HUGHES, T.J., "Mathematical Foundations of Elasticity", Prentice Hall, Englewood Clifts, N.J., 1983.

[3] STRUIK, D.J., "Lectures on Classical Differential Geometry", Addison Wesley, Reading, Mass, 1950.