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A Note Concerning Gauss-Jackson Method

Ana B. González and Pablo Martín

Depto. de Matemática Aplicada a la Ingeniería, E.T.S. de Ingenieros Industriales, Universidad de Valladolid, 47011-*Valladolid, Spain*

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1. Introduction

Specialized literature (Bate *et al.* [1], Fox [2], Herrick [4], Merson [5], Roy [6], etc.) concerning studies on Orbital Dynamics usually mentions the *Gauss-Jackson* or *sum squared* ($\sum^2$) method for the numerical integration of second order differential equations. However, as far as we know, no detailed description of this code is available and there is some confusion about the order of the method and its relation with the Störmer method. In this paper we present a simple way of deriving this algorithm and its corresponding analog for first order equations from the Störmer and Adams methods, respectively. We show that the Gauss-Jackson method can be conceived as a consequence of this, and therefore there is no difficulty in determining the order of the method. Finally, we obtain an initialization technique for its implementation, we show an advantage of it as compared with the traditional multistep methods when applied in PEC mode by suppressing the corrector stage in the intermediate steps.

2. Gauss-Jackson method for first order equations

Let us consider the initial value problem

$$\dot{y} = f(x, y)\,, \quad y(0) = y_0\,. \tag{1}$$

The Gauss-Jackson method for this problem admits the expression (see Merson [5])

$$y_{k+1} = h\left(\nabla^{-1} f_k + \sum_{j=0}^{l} F_j \nabla^j f_k\right), \tag{2}$$

where the coefficients F_j are obtained from the generating function

$$G_1(t) = -\frac{1}{(1-t)\log(1-t)} - \frac{1}{t}\,. \tag{3}$$

Let us consider now the Adams-Bashforth method of order $l+2$:

$$\nabla y_{k+1} = h\left(f_k + \sum_{j=1}^{l+1} \alpha_j \nabla^j f_k\right). \tag{4}$$

If the operator ∇^{-1} acts on both sides we have

$$y_{k+1} = h\left(\nabla^{-1} f_k + \sum_{j=1}^{l+1} \alpha_j \nabla^{j-1} f_k\right) = h\left(\nabla^{-1} f_k + \sum_{j=0}^{l} \alpha_{j+1} \nabla^j f_k\right). \tag{5}$$

Formula (5) is similar to (2). The generating function for the coefficients of the Adams-Bashforth method is (see Henrici [3])

$$G_2(t) = -\frac{t}{(1-t)\log(1-t)}\,. \tag{6}$$

By simple calculations we arrive at the relation

$$G_1(t) = \frac{G_2(t) - 1}{t}\,, \tag{7}$$

and after carrying out Taylor expansions we recognize that $F_j = \alpha_{j+1}$, and therefore the Gauss-Jackson method is a reformulation of the classical Adams-Bashforth code.

3. Gauss-Jackson method for second order equations

The Gauss-Jackson method for the initial value problem

$$\ddot{y} = f(x, y, \dot{y})\,, \quad y(0) = y_0\,, \ \dot{y}(0) = \dot{y}_0\,, \tag{8}$$

is given by the formulae [5]

$$\dot{y}_{k+1} = h\left(\nabla^{-1} f_k + \sum_{j=0}^{l} F_j \nabla^j f_k\right), \tag{9}$$

$$y_{k+1} = h^2\left(\nabla^{-2} f_k + \sum_{j=0}^{l} C_j \nabla^j f_k\right), \tag{10}$$

where the coefficients C_j are obtained from the generating function

$$G_3(t) = \frac{1}{(1-t)(\log(1-t))^2} - \frac{1}{t^2}. \tag{11}$$

As we have shown in the preceding section, Formula (9) corresponds to the Adams-Bashforth method, and we can prove that Formula (10) describes Störmer method.

Indeed, Störmer's formula is given by

$$\nabla^2 y_{k+1} = h^2 \left(f_k + \sum_{j=2}^{l+2} \beta_j \nabla^j f_k \right). \tag{12}$$

Letting the operator ∇^{-2} act on both sides we have

$$y_{k+1} = h^2 \left(\nabla^{-2} f_k + \sum_{j=0}^{l} \beta_{j+2} \nabla^j f_k \right). \tag{13}$$

The generating function for the Störmer method is

$$G_4(t) = \left(\frac{t}{\log(1-t)} \right)^2 \frac{1}{1-t}, \tag{14}$$

and proceeding in the same way as in Section 2, it is easily verified that

$$G_3(t) = \frac{G_4(t) - 1}{t^2}, \tag{15}$$

whence $C_j = \beta_{j+2}$, and the Gauss-Jackson method for second order equations can be conceived as a reformulation of the Störmer algorithm.

4. Initialization of Gauss-Jackson method

In order to implement the Gauss-Jackson code we have to know the values of $\nabla^{-1} f_k$ and $\nabla^{-2} f_k$. Let us suppose that we have computed the set of values $f_0, f_1, \dots f_{l+1}$ by a starting procedure. From these values we can construct the differences $\nabla^j f_{l+1}$, $j = 0, \dots, l+1$. From Equations (4) and (5), taking $k = l+1$, after subtracting we get

$$\nabla^{-1} f_{l+1} = \frac{1}{h} y_{l+1} + \sum_{j=0}^{l} (\alpha_j - \alpha_{j+1}) \nabla^j f_{l+1} + \alpha_{l+1} \nabla^{l+1} f_{l+1}. \tag{16}$$

		f_0								
			∇f_1							
		f_1		$\nabla^2 f_2$						
			∇f_2		$\nabla^3 f_3$					
		f_2		$\nabla^2 f_3$		$\nabla^4 f_4$				
			∇f_3		$\nabla^3 f_4$		$\nabla^5 f_5$			
		f_3		$\nabla^2 f_4$		$\nabla^4 f_5$		$\nabla^6 f_6$		
			∇f_4		$\nabla^3 f_5$		$\nabla^5 f_6$		$\nabla^7 f_7$	
		f_4		$\nabla^2 f_5$		$\nabla^4 f_6$		$\nabla^6 f_7$		$\nabla^8 f_8$
			∇f_5		$\nabla^3 f_6$		$\nabla^5 f_7$		$\nabla^7 f_8$	
		f_5		$\nabla^2 f_6$		$\nabla^4 f_7$		$\nabla^6 f_8$		
			∇f_6		$\nabla^3 f_7$		$\nabla^5 f_8$			
		f_6		$\nabla^2 f_7$		$\nabla^4 f_8$				
			∇f_7		$\nabla^3 f_8$					
		f_7		$\nabla^2 f_8$						
			∇f_8							
$\nabla^{-2} f_7$		f_8								
	$\nabla^{-1} f_8$									
$\nabla^{-2} f_8$										

Table 1. Table of differences.

For $k > l+1$, the values of $\nabla^{-1} f_k$ are easily calculated: $\nabla^{-1} f_k = \nabla^{-1} f_{k-1} + f_k$.

We can determine $\nabla^{-2} f_l$ in a similar way. By combining Formula (12) with Formula (13), we obtain

$$\nabla^{-2} f_l = \frac{1}{h^2} y_{l+1} + \sum_{j=0}^{l} (\beta_{j+1} - \beta_{j+2}) \nabla^j f_{l+1} + \beta_{l+2} \nabla^{l+1} f_{l+1} \,. \tag{17}$$

With Formula (17) we can get the initial value of $\nabla^{-2} f_l$. And for $k > l$, the values of $\nabla^{-2} f_k$ are given by $\nabla^{-2} f_k = \nabla^{-2} f_{k-1} + \nabla^{-1} f_k$.

In Table 1 we show the set of differences required to initializate the method for l=7. The values $\nabla^j f_k$, $j > 0$ are calculated using the fact that $\nabla^j f_{k+1} - \nabla^j f_k = \nabla^{j+1} f_{k+1}$. The values $\nabla^{-1} f_8$ and $\nabla^{-2} f_7$ are determined with Equations (16) and (17), respectively, The table presented by Merson [5] differs from ours (Table 1) in that he had to resort to a previous knowledge of an additional set of differences.

5. Gauss-Jackson method in PEC mode

In this section we will restrict ourselves to considering the performance of the method for first order equations. The results are similar for second order equations.

The implicit Gauss-Jackson method obeys the formula

$$y_{k+1} = h\left(\nabla^{-1} f_{k+1} + \sum_{j=0}^{l} (\alpha_{j+1} - \alpha_j)\nabla^j f_{k+1}\right) \tag{18}$$

$$= h\left(\nabla^{-1} f_k + \sum_{j=0}^{l-1} \alpha_{j+1}\nabla^j f_k\right) + h\alpha_{l+1}\nabla^l f_{k+1}\,. \tag{19}$$

With the help of this expression we can write the implicit method in PEC mode. The predictor stage will be the explicit Gauss-Jackson method of order $l+1$,

$$\tilde{y}_{k+1} = h\left(\nabla^{-1} f_k + \sum_{j=0}^{l-1} \alpha_{j+1}\nabla^j f_k\right), \tag{20}$$

the evaluation stage employs

$$f_{k+1} = f(x_{k+1}, \tilde{y}_{k+1}), \tag{21}$$

and the corrector produces

$$y_{k+1} = \tilde{y}_{k+1} + h\alpha_{l+1}\nabla^l f_{k+1}\,. \tag{22}$$

We realize that the next step does not require the value of y_{k+1}, and so we can avoid the corrector stage (22) at the intermediate steps and only use it at the point at which we seek the approximation to the solution.

This conclusion represents an advantage over the Adams form of the methods, because it has not any analog when we write the method in this last form. Adams-Moulton method has the expression

$$y_{k+1} = y_k + h\sum_{j=0}^{l+1} (\alpha_j - \alpha_{j-1})\nabla^j f_{k+1} \tag{23}$$

$$= y_k + h\sum_{j=0}^{l} \alpha_j\nabla^j f_k + h\alpha_{l+1}\nabla^{l+1} f_{k+1}\,. \tag{24}$$

Maintaining the above notations, we have the three following stages:

$$\tilde{y}_{k+1} = y_k + h\sum_{j=0}^{l}\alpha_j\nabla^j f_k\,, \tag{25}$$

$$f_{k+1} = f(x_{k+1}, \tilde{y}_{k+1})\,, \tag{26}$$

$$y_{k+1} = \tilde{y}_{k+1} + h\alpha_{l+1}\nabla^{l+1} f_{k+1}\,. \tag{27}$$

With this formulation we can not suppress the corrector stage, because we need the value of y_{k+1} for the next step.

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