

Chern-Dold Character in Elliptic Cohomology

ABDELAZIZ KHELDOUNI AND FAYÇAL LAMRINI

*Université Med Benabdallah, Dépt. de Maths., Fac des Sciences Dhar-Mehraz,
B.P. 1796 Fès-Atlas, Maroc*

(Presented by A.M. Cegarra)

AMS Subject Class. (1991): 55N22

Received May 30, 1995

INTRODUCTION

In [7] L. Smith defines and studies the Todd character th , which is the composition of the Chern character and the classifying map of K -theory as a complex oriented cohomology theory, $th : MU \rightarrow K \rightarrow \mathcal{H}(\pi_*(K) \otimes \mathbb{Q})$ where $\pi_*(K) = \mathbb{Z}[u, u^{-1}]$. For any CW-complex X , we have:

$$th : MU_n(X) \longrightarrow \sum_{k=0}^{[n/2]} H_{n-2k}(X, \mathbb{Q})[u]$$

defined for any $x \in H^*(X, \mathbb{Q})$ and $\alpha = [M, f] \in MU_*(X)$, by

$$\langle x, th(\alpha) \rangle = \langle f^*(x)Td(M), [M] \rangle,$$

$[M]$ being the fundamental homology class of M and \langle, \rangle the Kronecker product. L. Smith proved the following integrality theorem (see [7], Theorem 2.1): Let X be a CW-complex and $\alpha \in MU_{2n}(X)$, then $\mu_{n-k}th_{2k}(\alpha)$ is integral (i.e. it belongs to $\text{Im}\{H_{2k}(X, \mathbb{Z}) \rightarrow H_{2k}(X, \mathbb{Q})\}$), where $\mu_t = \prod_{\text{primes}} p^{[t/p-1]}$. (A similar result holds for odd-dimensional classes). If we consider elliptic cohomology, instead of K -theory, we can define the elliptic Chern-Dold character, what we will call *Elliptic Character*,

$$e : MU_*(X) \rightarrow H_*(X, \pi_*(\mathcal{E}ll) \otimes \mathbb{Q})$$

where the ring $\pi_*(\mathcal{E}ll) = \mathbb{Z}[\frac{1}{2}][\delta, \varepsilon, (\delta^2 - \varepsilon)^{-1}]$, is $4\mathbb{Z}$ -graded.

In this paper we define elliptic polynomials associated to elliptic genus (as Todd polynomials are to Todd genus), and define the elliptic character. Then we derive an integrality result of this character analogously of that of Todd character.

THEOREM 0.1. *Let X a CW-complex and $\alpha \in MU_n(X)$,*

$$e(\alpha) = \sum_{0 \leq 4k \leq n} e_{n-4k}(\alpha) \in \sum_{0 \leq 4k \leq n} H_{n-4k}(X, \pi_{4k}(\mathcal{E}ll) \otimes \mathbb{Q})$$

Then $\mu_{2k}e_{n-4k}(\alpha)$ is integral, i.e, it belongs to

$$\text{Im} \{H_{n-4k}(X; \pi_{4k}(\mathcal{E}ll)) \rightarrow H_{n-4k}(X, \pi_{4k}(\mathcal{E}ll) \otimes \mathbb{Q})\}.$$

THEOREM 0.2. *Let X a CW-complex and $\alpha \in MU_n(X)$. Then for an odd prime p , $p^{[2k/p-1]}e_{n-4k}(\alpha)$ is p -integral, i.e., it belongs to*

$$\text{Im} \{H_{n-4k}(X; \pi_{4k}(\mathcal{E}ll) \otimes \mathbb{Z}_{(p)}) \rightarrow H_{n-4k}(X, \pi_{4k}(\mathcal{E}ll) \otimes \mathbb{Q})\}$$

We end this introduction by noting some remarks:

Remarks. a) In the same way L. Smith establishes a connection between Todd character and Landweber-Novikov operations (cf. [8]), we give a similar expression of elliptic character. We described this in “Caractère elliptique et opérations cohomologiques” (to appear).

b) Actually, the integrality results of the Todd character and elliptic character can be derived from the integrality results for Todd polynomials $Td_n(c_1, \dots, c_n)$, and elliptic polynomials $E_n(p_1, \dots, p_n)$ respectively, which are special cases of more general situation concerning universal polynomials $U_n(c_1, \dots, c_n)$ (cf. [5]).

We wish to thank the referee for critical reading of an earlier version of this manuscript and for several suggestions that have improved the exposition.

1. ELLIPTIC COHOMOLOGY AND ELLIPTIC GENUS

By elliptic genus is meant a ring homomorphism $\Phi : MU_* \rightarrow A$, from the complex bordism group MU_* to a \mathbb{Q} -algebra A , given by its logarithm $g(X)$ which is a formal power series defined by a first kind elliptic integral:

$$g(X) = \int_0^X \frac{dt}{\sqrt{1 - 2\delta t^2 + \epsilon t^4}} \tag{1}$$

where $\delta, \epsilon \in A$. Making use of the binomical expansion of (1) one easily see that all coefficients lie in $\mathbb{Z}[1/2][\delta, \epsilon]$. Another expression of $g(X)$ is given by means of the Legendre polynomials (see [6]),

$$g(X) = \sum_{n=0} P_n(\delta/\sqrt{\epsilon}) \frac{\sqrt{\epsilon}^n}{2n+1} X^{2n+1} \tag{2}$$

According to [6], if δ and ε are algebraically independent, the Landweber functor exact theorem holds. We then obtain a new cohomology theory after localization either by $\gamma = \varepsilon, \delta^2 - \varepsilon$ or $\gamma = \Delta = \varepsilon(\delta^2 - \varepsilon)^2$. We shall adopt the same notation for these three cohomology theories, namely $\mathcal{E}l\mathcal{U}^*(X) = MU^*(X) \otimes_{MU^*} \mathbb{Z}[1/2][\delta, \varepsilon, \gamma^{-1}]$. These are complex oriented cohomology theories, their formal group law being the Euler's one,

$$F_E(X, Y) = \frac{X\sqrt{1 - 2\delta Y^2 + \varepsilon Y^4} + Y\sqrt{1 - 2\delta X^2 + \varepsilon X^4}}{1 - \varepsilon X^2 Y^2}$$

Remark 1.1. The coefficients group $\mathbb{Z}[1/2][\delta, \varepsilon, \gamma^{-1}]$ has necessarily the graduation given by $|\delta| = 4, |\varepsilon| = 8$. It follows that $\mathbb{Z}[1/2][\delta, \varepsilon, \gamma^{-1}]$ has $4\mathbb{Z}$ -graduation. In fact if A is a graded \mathbb{Q} -algebra and $g(X) = X + \sum_{i \geq 1} m_i X^{i+1} \in A[[X]]$, then a necessary condition for $g(X)$ to be the logarithm of a formal group law $F(X, Y) = X + Y + \sum_{i, j \geq 1} a_{ij} X^i Y^j$ (with $|a_{ij}| = -2(i + j - 1)$), associated to a complex oriented cohomology theory is: $|m_i| = -2i$. Indeed, from $F(X, Y) = g^{-1}(g(X) + g(Y))$ we get,

$$g'(X) = \frac{1}{\partial_1 F(0, X)}$$

or

$$1 + \sum_{i=1} (i + 1)m_i X^i = (1 + \sum_{i=1} a_{1i} X^i)^{-1}.$$

For $k = 1$, we have $\sum_{i+j=k} a_{1i}(1 + j)m_j = 0$, then $|m_k| = |a_{1k}| = -2k$. For the elliptic cohomology, we have

$$g(X) = \sum_{n \geq 0} \frac{P_n(\delta/\sqrt{\varepsilon})\sqrt{\varepsilon}^n}{2n + 1} X^{2n+1} = \sum_{k \geq 0} m_k X^{k+1}.$$

Then $m_{2n+1} = 0, m_{2n} = \frac{P_n(\delta/\sqrt{\varepsilon})\sqrt{\varepsilon}^n}{2n+1}$ and $d^0(P_n(\delta/\sqrt{\varepsilon})\sqrt{\varepsilon}^n) = 4n$.

If $k = 1$, then $P_1(\delta/\sqrt{\varepsilon})\sqrt{\varepsilon} = \delta$ and $|\delta| = 4$

If $k = 2$, then $P_2(\delta/\sqrt{\varepsilon})\varepsilon = \frac{1}{2}(3\delta^2 - \varepsilon)$ and $|\varepsilon| = 8$.

The following proposition determines the coefficient group of the elliptic cohomology.

PROPOSITION 1.2. *If $\gamma = \delta^2 - \varepsilon$ and $A = \mathbb{Z}[1/2][\delta, \varepsilon, \gamma^{-1}]$, then the graded $\mathbb{Z}[1/2]$ -algebra A is completely determined by γ, δ and $A_0 = \mathbb{Z}[1/2][\delta^2/\gamma]$. For $k \in \mathbb{Z}$, we have $A_{8k} = \gamma^k A_0$, and $A_{8k+4} = \gamma^k \delta A_0 = \gamma^k A_4$.*

Proof. As A is obtained after localizing the $4\mathbb{Z}$ -graded ring $\mathbb{Z}[1/2][\delta, \varepsilon]$ by the eight dimensional element γ , it is completely determined by $A_0 = \varinjlim_k \gamma^{-k}\mathbb{Z}[1/2][\delta, \varepsilon]_{8k}$ and $A_4 = \varinjlim_k \gamma^{-k}\mathbb{Z}[1/2][\delta, \varepsilon]_{8k+4}$. The components $\mathbb{Z}[1/2][\delta, \varepsilon]_{8k}$ and $\mathbb{Z}[1/2][\delta, \varepsilon]_{8k+4}$ are free $\mathbb{Z}[1/2]$ -modules with basis $B_0 = \{\delta^{2k}, \delta^{2k-2}\varepsilon, \dots, \delta^2\varepsilon^{k-1}, \varepsilon^k\}$ and $B_4 = \{\delta^{2k+1}, \delta^{2k-1}\varepsilon, \dots, \delta^3\varepsilon^{k-1}, \delta\varepsilon^k\}$ respectively. Let $M_{k+1} = (a_{ij})$ be the triangular matrix of order $k + 1$ defined by,

$$a_{ij} = (-1)^{i-1}C_{i-1}^{j-1} = (-1)^{i-1} \frac{(j-1)!}{(i-1)!(j-i)!}, \quad i \leq j$$

We get the basis

$$M_{k+1}B_0 = \{\delta^{2k}, \delta^{2k-2}\gamma, \delta^{2k-4}\gamma^2, \dots, \delta^2\gamma^{k-1}, \gamma^k\}$$

and

$$M_{k+1}B_4 = \{\delta^{2k+1}, \delta^{2k-1}\gamma, \dots, \delta^3\gamma^{k-1}, \delta\gamma^k\}.$$

Then $\gamma^{-k}\mathbb{Z}[1/2][\delta, \varepsilon]_{8k}$ and $\gamma^{-k}\mathbb{Z}[1/2][\delta, \varepsilon]_{8k+4}$ becomes free $\mathbb{Z}[1/2]$ -modules with basis

$$\{\delta^{2k}/\gamma^k, \delta^{2k-2}/\gamma^{k-1}, \dots, \delta^2/\gamma, 1\} \text{ and } \{\delta^{2k+1}/\gamma^k, \delta^{2k-1}/\gamma^{k-1}, \dots, \delta^3/\gamma, \delta\}$$

respectively. Therefore $A_0 = \mathbb{Z}[1/2][\delta^2/\gamma]$ and $A_4 = \delta A_0$. ■

Remark 1.3. We get a similar result if we localise by $\gamma = \varepsilon$ instead of $\gamma = \delta^2 - \varepsilon$. The algebra $\mathbb{Z}[1/2][\delta, \varepsilon, \varepsilon^{-1}]$ is determined by $\mathbb{Z}[1/2][\delta^2/\varepsilon]$ and $\varepsilon\mathbb{Z}[1/2][\delta^2/\varepsilon]$.

2. ELLIPTIC POLYNOMIALS

According to [9], the characteristic power series of the universal elliptic genus is,

$$P(X) = 1 - \sum_{k>0} \frac{G_{2k}^*}{2^{2k-2}(2k-1)!} X^{2k}$$

where G_{2k}^* , $k \in \mathbb{N}$, are modular functions for the subgroup $\Gamma_0(2)$ of the modular group $SL_2(\mathbb{Z})$, consisting of all matrix $\begin{pmatrix} a & b \\ 2c & d \end{pmatrix}$.

$$G_{2k}^*(\tau) = G_{2k}(\tau) - 2^{2k-2}G_{2k}(2\tau)$$

with $G_{2k}(\tau) = \sum_{n,m \in \mathbb{Z}^*} \frac{1}{(n\tau+m)^{2k}}$ is the k^{th} Eisenstein series, $\tau \in \mathcal{H} = \{z \in \mathbb{C}/\text{Im } z > 0\}$, (see [9]).

Let us denote $\tilde{P}(X^2) = P(X)$. The following definition is similar to that of [4].

DEFINITION 2.1. The elliptic polynomial $E_n(p_1, \dots, p_n)$ is the coefficient of X^n in the product $\prod_{1 \leq i \leq n} \tilde{P}(\mu_i X)$, where $p_i = \sigma(\mu_1, \dots, \mu_n)$ denotes the i^{th} elementary symmetric function of μ_1, \dots, μ_n .

Before giving the expression of $E_n(p_1, \dots, p_n)$, we shall adopt the following notations:

For each integer n , $\mathcal{P}(n)$ is the set of all partitions $\omega = (i_1, \dots, i_k)$ of n . For each ω of $\mathcal{P}(n)$ we denote by $|\omega|$ the cardinal of ω .

For $t \in \{1, \dots, n\}$, $r(t)$ is the number of elements of ω identical to t . We have,

$$\sum_{t=1}^n t.r(t) = \sum_{j=1}^k i_j$$

Let $d(\omega) = \prod_{1 \leq t \leq n} r(t)!$, μ^ω the symmetrical expression $\sum \mu_1^{i_1} \dots \mu_k^{i_k}$, and $H_\omega = H_{i_1} \dots H_{i_k}$, with $H_k = \frac{G_{2k}^*}{2^{2k-2}(2k-1)!}$,

PROPOSITION 2.2. We have :

$$E_n(p_1, \dots, p_n) = \sum_{k=1}^n (-1)^k \sum_{\substack{\omega \in \mathcal{P}(n) \\ |\omega|=k}} \frac{H_\omega}{d(\omega)}.$$

Proof. It suffices to apply the definition 2.1 ■

Remark 2.3. If $S_t = \sum_{i=1}^{i=n} \mu_i^t$, then $S_\omega = \prod_{t \in \omega} S_t$ where $\omega = (i_1, \dots, i_k) \in \mathcal{P}(n)$. Making use of the Newton formula connecting S_t and the elementary symmetrical polynomials σ_t , we get the formula,

$$\mu^\omega = S_\omega = \sum_{t=2}^{|\omega|=k} (-1)^{t-1} (t-1)! \sum_{\substack{\theta \subset \omega \\ |\theta|=t}} S_{t(\theta)} S_{\omega-\theta}$$

This formula enables us to get the small polynomials of the sequence (E_n) ,

$$E_1(p_1) = -H_1 p_1.$$

$$E_2(p_1, p_2) = (H_1^2 + 2H_2)p_2 - H_2 p_1^2.$$

$$E_3(p_1, p_2, p_3) = -H_3 p_1^3 + (H_1 H_2 + 3H_3)p_1 p_2 - (H_1^3 + 3H_1 H_2 + H_3)p_3.$$

Or alternatively:

$$E_1(p_1) = -G_2^* p_1.$$

$$E_2(p_1, p_2) = -\frac{G_4^*}{24} p_1^2 + ((G_2^*)^2 + \frac{2G_4^*}{24}) p_2$$

$$E_3(p_1, p_2, p_3) = -\frac{G_6^*}{24 \cdot 5!} p_1^3 + (\frac{G_2^* G_4^*}{3!2^2} + \frac{3G_6^*}{5!2^4}) p_1 p_2 - ((G_2^*)^3 + \frac{G_2^* G_4^*}{8} + \frac{G_6^*}{5!2^4}) p_3.$$

PROPOSITION 2.4. *We have,*

$$\delta = -3G_2^*$$

$$\varepsilon = 7G_2^* + \frac{5}{12}G_4^*$$

$$\gamma = \delta^2 - \varepsilon = 2G_2^* - \frac{5}{12}G_4^*$$

$$\mathbb{Z}[1/2][\delta, \varepsilon, \gamma^{-1}] = \mathbb{Z}[1/6][G_2^*, G_4^*, \gamma^{-1}].$$

Proof. The elliptic genus is characterised by its logarithm g given by (2). If $P(X)$ denotes its characteristic power series, we have $P(X) = \frac{X}{g^{-1}(X)}$. $g(X)$ having the form $\sum_{i \geq 0} a_{2i} X^{2i+1}$, then: $P(X) = 1 + a_2 X^2 + (a_4 - 2a_2^2) X^4 + \dots$. Since $a_2 = \frac{P_1(\delta/\sqrt{\varepsilon})\sqrt{\varepsilon}}{3} = -G_2^*$, we obtain $\delta = -3G_2^*$. Similarly, as $a_4 - 2a_2^2 = -\frac{G_4^*}{3!2^2} = \frac{P_2(\delta/\sqrt{\varepsilon})\sqrt{\varepsilon}^2}{5} - \frac{2}{9}P_1(\delta/\sqrt{\varepsilon})^2\varepsilon = \frac{1}{10}(3\delta^2 - \varepsilon) - \frac{2}{9}\delta^2$, we have $\varepsilon = 7(G_2^*)^2 + \frac{5}{12}G_4^*$. ■

The elliptic polynomials $E_n(p_1, \dots, p_n)$ are homogenous polynomials in p_1, \dots, p_n of weight n (each p_i having weight i), with coefficients in $\mathbb{Q}[\delta, \varepsilon]$. For example:

$$E_1(p_1) = \frac{\delta}{3} p_1.$$

$$E_2(p_1, p_2) = (\frac{7}{90} p_1^2 - \frac{2}{45} p_2) \delta^2 + (p_2 - \frac{1}{2} p_1^2) \frac{\varepsilon}{5}$$

$$E_3(p_1, p_2, p_3) = [\frac{p_3}{27} + \frac{31}{1890}(p_1^3 - 3p_1 p_2 - 3p_3) + \frac{7}{270}(p_1 p_2 + 3p_3)] \delta^3 - [\frac{1}{270}(p_1 p_2 + 3p_3) + \frac{27}{1890}(p_1^3 - 3p_1 p_2 - 3p_3)] \delta \varepsilon.$$

According to [5], if $\mu_{2n} = \prod_{3 \leq q \leq 2n+1} q^{[2n/(q-1)]}$, we have,

PROPOSITION 2.5. *The polynomial $\mu_{2n} E_n(p_1, \dots, p_n)$ is homogeneous in p_1, \dots, p_n of degree n , its coefficients are homogeneous polynomials in δ, ε , with coefficients in $\mathbb{Z}[1/2]$.*

3. ELLIPTIC CHARACTER

DEFINITION 3.1. Let M be a differentiable manifold. The n^{th} elliptic class of M is $E_n(p_1(M), \dots, p_n(M))$ where $p_i(M)$ is the i^{th} Pontrjagin class of M . $E(M) = \sum_n E_n(p_1(M), \dots, p_n(M))$ is the total elliptic class of M .

Let us denote $\mathcal{A} = \mathbb{Q}[\delta, \varepsilon, \gamma^{-1}]$. We define the elliptic character by the homology transformation :

$$e : MU_n(X) \longrightarrow \mathcal{H}_n(X, \mathcal{A}) = \sum_{0 \leq 4i < n} H_{n-4i}(X, \mathcal{A}_{4i})$$

where X is a CW-complex. To give an explicit expression for e , we recall that an homological class $x \in H_*(X, \mathbb{Q})$ is uniquely determined by the mapping $\langle, \rangle : H^*(X, \mathbb{Q}) \rightarrow \mathbb{Q}$, given by the Kronecker product. Suppose that $\alpha = [M, f] \in MU_*(X)$, then $e(\alpha)$ is determined by the formula,

$$\langle x, e(\alpha) \rangle = \langle f^*(x)E(M), [M] \rangle \tag{3}$$

where x is an arbitrary element of $H^*(X, \mathcal{A})$, $E(M)$ is the total elliptic class of M (Definition 3.1) and $[M]$ is the fundamental homology class of M . For X a point, $e : MU_* \rightarrow \mathcal{A}$ can be identified with the elliptic genus.

If $\alpha = [M, f] \in MU_n(X)$ is an homogenous element, the elliptic character has $e_{n-4i}(\alpha) \in H_{n-4i}(X, \mathcal{A}_{4i})$ for components and since by (3) $e_m(\alpha) = 0$ for $m > n$, we have,

$$e(\alpha) = \sum_{0 \leq 4i \leq n} e_{n-4i}(\alpha).$$

Furthermore $e_n(\alpha) = \mu(\alpha)$ where μ is the classical Thom homomorphism (cf. [2]). From proposition 2.5 we can state,

THEOREM 3.2. Let X a CW-complex and $\alpha \in MU_n(X)$,

$$e(\alpha) = \sum_{0 \leq 4k \leq n} e_{n-4k}(\alpha), \in \sum_{0 \leq 4k \leq n} H_{n-4k}(X, \pi_{4k}(\mathcal{E}ll) \otimes \mathbb{Q})$$

Then $\mu_{2k}e_{n-4k}(\alpha)$ is integral, i.e, it belongs to

$$\text{Im} \{ H_{n-4k}(X; \pi_{4k}(\mathcal{E}ll)) \rightarrow H_{n-4k}(X, \pi_{4k}(\mathcal{E}ll) \otimes \mathbb{Q}) \}.$$

Proof. Let $x \in H_{n-4k}(X, \mathcal{A}_{4k})$. We have,

$$\begin{aligned} \langle x, \mu_{2k}e_{n-4k}(\alpha) \rangle &= \langle \mu_{2k}f^*(x)E_k(M), [M] \rangle \\ &= \langle f^*(x)\mu_{2k}E_k(M), [M] \rangle \in \pi_{4k}(\mathcal{E}ll). \end{aligned}$$

■

Remark 3.3. Let p be a prime, and denote by $\mathbb{Z}_{(p)}$ the ring of integers localized at p . According to [7], we say that an homological class $x \in H_*(X, \mathbb{Q})$ is p -integral if it belongs to $\text{Im} \{H_*(X, \mathbb{Z}_{(p)}) \rightarrow H_*(X, \mathbb{Q})\}$.

We can now give a p -integrality theorem for the elliptic character as follows,

THEOREM 3.4. *Let X a CW-complex and $\alpha \in MU_n(X)$. Then for an odd prime p , $p^{\lfloor 2k/p-1 \rfloor} e_{n-4k}(\alpha)$ is p -integral (i.e it belongs to $\text{Im} \{H_{n-4k}(X; \pi_{4k}(\mathcal{E}ll) \otimes \mathbb{Z}_{(p)}) \rightarrow H_{n-4k}(X, \pi_{4k}(\mathcal{E}ll) \otimes \mathbb{Q})\}$).*

Proof. The proof is immediate from theorem 3.2. ■

4. CONCLUSION

There are some naive questions arising from the reading of [3]:

- a) Can we associate some topological invariant to elliptic character, in the same way that Adams-Toda e_c invariant is associated to Todd character?
 - b) What are the spin-off of the elliptic character on $\text{hom.dim}_{MU_*}(MU_*(X))$?
- This will be the subject of a next paper.

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